Pearson Edexcel Level 3 Advanced GCE in Mathematics (9MA0)

Two-year Scheme of Work

For first teaching from September 2017

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Introduction

Content of this document is based on the accredited version of Pearson Edexcel Level 3 Advanced GCE in Mathematics (9MA0) and includes detailed scheme of work for pure and applied content of AS and A level Mathematics.

This scheme of work is based upon a two-year delivery model for A Level Mathematics where all AS Mathematics content is being taught in the first year (to aid co-teaching of AS and A level groups together) and the remaining A level Mathematics content in the second year.

It can be used directly as a scheme of work for the A level Mathematics specification (9MA0).

The scheme of work is broken up into units and sub-units, so that there is greater flexibility for moving topics around to meet planning needs.

Each unit contains:

- Specification references
- Prior knowledge
- Keywords
- Notes.

Each sub-unit contains:

- Recommended teaching time, though of course this is adaptable according to individual teaching needs
- Objectives for students at the end of the sub-unit
- Teaching points
- Opportunities for problem-solving and modelling
- Common misconceptions and examiner report quotes (from legacy Specifications)
- Notes

Teachers should be aware that the estimated teaching hours are approximate and should be used as a guideline only.

Our free support for the AS and A level Mathematics specifications can be found on the Pearson Edexcel Mathematics website (quals.pearson.com/Alevelmaths2017) and on the Emporium (www.edexcelmaths.com).

AS level Mathematics
Paper 1: Pure Mathematics 62.5%, 2 hours, 100 marks
Paper 2: Statistics and Mechanics 37.5%, 1 hour 15 minutes, 60 marks

A level Mathematics	
Paper 1: Pure Mathematics 33%, 2 hours, 100 marks	Any pure content can be assessed on either paper
Paper 2: Pure Mathematics 33%, 2 hours, 100 marks	
Paper 3: Statistics and Mechanics 33%, 2 hours, 100 marks	Section A: Statistics (50 marks) Section B: Mechanics (50 marks)

Year 1: AS Mathematics pure content

Pure Mathematics

Unit		Title	Estimated hours
1		**Algebra and functions**	
	a	Algebraic expressions – basic algebraic manipulation, indices and surds	**4**
	b	Quadratic functions – factorising, solving, graphs and the discriminants	**4**
	c	Equations – quadratic/linear simultaneous	**4**
	d	Inequalities – linear and quadratic (including graphical solutions)	**5**
	e	Graphs – cubic, quartic and reciprocal	**5**
	f	Transformations – transforming graphs – f(x) notation	**5**
2		**Coordinate geometry in the (x, y) plane**	
	a	Straight-line graphs, parallel/perpendicular, length and area problems	**6**
	b	Circles – equation of a circle, geometric problems on a grid	**7**
3		**Further algebra**	
	a	Algebraic division, factor theorem and proof	**8**
	b	The binomial expansion	**7**
4		**Trigonometry**	
	a	Trigonometric ratios and graphs	**6**
	b	Trigonometric identities and equations	**10**
5		**Vectors (2D)**	
	a	Definitions, magnitude/direction, addition and scalar multiplication	**7**
	b	Position vectors, distance between two points, geometric problems	**7**
6		**Differentiation**	
	a	Definition, differentiating polynomials, second derivatives	**6**
	b	Gradients, tangents, normals, maxima and minima	**6**
7		**Integration**	
	a	Definition as opposite of differentiation, indefinite integrals of x^n	**6**
	b	Definite integrals and areas under curves	**5**
8		**Exponentials and logarithms:** Exponential functions and natural logarithms	**12**
			120 hours

Year 1: AS Mathematics applied content
Statistics and Mechanics

Unit		Title	Estimated hours
Section A – Statistics			
1		**Statistical sampling**	
	a	Introduction to sampling terminology; Advantages and disadvantages of sampling	**1**
	b	Understand and use sampling techniques; Compare sampling techniques in context	**2**
2		**Data presentation and interpretation**	
	a	Calculation and interpretation of measures of location; Calculation and interpretation of measures of variation; Understand and use coding	**4**
	b	Interpret diagrams for single-variable data; Interpret scatter diagrams and regression lines; Recognise and interpret outliers; Draw simple conclusions from statistical problems	**8**
3		**Probability:** Mutually exclusive events; Independent events	**3**
4		**Statistical distributions:** Use discrete distributions to model real-world situations; Identify the discrete uniform distribution; Calculate probabilities using the binomial distribution (calculator use expected)	**5**
5		**Statistical hypothesis testing**	
	a	Language of hypothesis testing; Significance levels	**2**
	b	Carry out hypothesis tests involving the binomial distribution	**5**
			30 hours
Section B – Mechanics			
6		**Quantities and units in mechanics**	
	a	Introduction to mathematical modelling and standard S.I. units of length, time and mass	**1**
	b	Definitions of force, velocity, speed, acceleration and weight and displacement; Vector and scalar quantities	**2**
7		**Kinematics 1 (constant acceleration)**	
	a	Graphical representation of velocity, acceleration and displacement	**4**
	b	Motion in a straight line under constant acceleration; *suvat* formulae for constant acceleration; Vertical motion under gravity	**6**
8		**Forces & Newton's laws**	
	a	Newton's first law, force diagrams, equilibrium, introduction to $\mathbf{i}$, $\mathbf{j}$ system	**4**
	b	Newton's second law, '$F = ma$', connected particles (no resolving forces or use of $F = \mu R$); Newton's third law: equilibrium, problems involving smooth pulleys	**6**
9		**Kinematics 2 (variable acceleration)**	
	a	Variable force; Calculus to determine rates of change for kinematics	**4**
	b	Use of integration for kinematics problems i.e. $r = \int v \, dt$, $v = \int a \, dt$	**3**
			30 hours

Year 2: Remaining A Level Mathematics pure content

Pure Mathematics

Unit		Title	Estimated hours
1		**Proof:** Examples including proof by deduction* and proof by contradiction	3
2		**Algebraic and partial fractions**	
	a	Simplifying algebraic fractions	2
	b	Partial fractions	3
3		**Functions and modelling**	
	a	Modulus function	2
	b	Composite and inverse functions	3
	c	Transformations	3
	d	Modelling with functions* *examples may be Trigonometric, exponential, reciprocal etc.	2
4		**Series and sequences**	
	a	Arithmetic and geometric progressions (proofs of ‘sum formulae’)	4
	b	Sigma notation	2
	c	Recurrence and iterations	3
5		**The binomial theorem**	
	a	Expanding $(a + bx)^n$ for rational n; knowledge of range of validity	4
	b	Expansion of functions by first using partial fractions	3
6		**Trigonometry**	
	a	Radians (exact values), arcs and sectors	4
	b	Small angles	2
	c	Secant, cosecant and cotangent (definitions, identities and graphs); Inverse trigonometrical functions; Inverse trigonometrical functions	3
	d	Compound* and double (and half) angle formulae *geometric proofs expected	6
	e	$R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$	3
	f	Proving trigonometric identities	4
	g	Solving problems in context (e.g. mechanics)	2
7		**Parametric equations**	
	a	Definition and converting between parametric and Cartesian forms	3
	b	Curve sketching and modelling	2

Unit		Title	Estimated hours
8		**Differentiation**	
	a	Differentiating sin x and cos x from first principles	**2**
	b	Differentiating exponentials and logarithms	**3**
	c	Differentiating products, quotients, implicit and parametric functions.	**6**
	d	Second derivatives (rates of change of gradient, inflections)	**2**
	e	Rates of change problems* (including growth and kinematics) *see Integration (part 2) – Differential equations	**3**
9		**Numerical methods***	
	a	Location of roots	**1**
	b	Solving by iterative methods (knowledge of 'staircase and cobweb' diagrams)	**3**
	c	Newton-Raphson method	**2**
	d	Problem solving	**2**
		***See Integration (part 2) for the trapezium rule	
10		**Integration (part 1)**	
	a	Integrating x^n (including when $n = –1$), exponentials and trigonometric functions	**4**
	b	Using the reverse of differentiation, and using trigonometric identities to manipulate integrals	**5**
11		**Integration (part 2)**	
	a	Integration by substitution	**4**
	b	Integration by parts	**3**
	c	Use of partial fractions	**2**
	d	Areas under graphs or between two curves, including understanding the area is the limit of a sum (using sigma notation)	**4**
	e	The trapezium rule	**2**
	f	Differential equations (including knowledge of the family of solution curves)	**4**
12		**Vectors (3D):** Use of vectors in three dimensions; knowledge of column vectors and **i**, **j** and **k** unit vectors	**5**
			120 hours

Year 2: Remaining A Level Mathematics applied content

Statistics and Mechanics

Unit		Title	Estimated hours
Section A – Statistics			
1		**Regression and correlation**	
	a	Change of variable	**2**
	b	Correlation coefficients Statistical hypothesis testing for zero correlation	**5**
2		**Probability**	
	a	Using set notation for probability Conditional probability	**5**
	b	Questioning assumptions in probability	**2**
3		**The Normal distribution**	
	a	Understand and use the Normal distribution	**5**
	b	Use the Normal distribution as an approximation to the binomial distribution Selecting the appropriate distribution	**5**
	c	Statistical hypothesis testing for the mean of the Normal distribution	**6**
			30 hours
Section B – Mechanics			
4		**Moments:** Forces' turning effect	**5**
5		**Forces at any angle**	
	a	Resolving forces	**3**
	b	Friction forces (including coefficient of friction μ)	**3**
6		**Applications of kinematics:** Projectiles	**5**
7		**Applications of forces**	
	a	Equilibrium and statics of a particle (including ladder problems)	**4**
	b	Dynamics of a particle	**4**
8		**Further kinematics**	
	a	Constant acceleration (equations of motion in 2D; the **i**, **j** system)	**3**
	b	Variable acceleration (use of calculus and finding vectors $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ at a given time)	**3**
			30 hours

Year 1: AS Mathematics pure content

Pure Mathematics

Unit		Title	Estimated hours
1		**Algebra and functions**	
	a	Algebraic expressions – basic algebraic manipulation, indices and surds	**4**
	b	Quadratic functions – factorising, solving, graphs and the discriminants	**4**
	c	Equations – quadratic/linear simultaneous	**4**
	d	Inequalities – linear and quadratic (including graphical solutions)	**5**
	e	Graphs – cubic, quartic and reciprocal	**5**
	f	Transformations – transforming graphs – f(x) notation	**5**
2		**Coordinate geometry in the (x, y) plane**	
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7		**Integration**	
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8		**Exponentials and logarithms:** Exponential functions and natural logarithms	**12**
			120 hours

UNIT 1: Algebra and Functions

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SPECIFICATION REFERENCES

2.1 Understand and use the laws of indices for all rational exponents

2.2 Use and manipulate surds, including rationalising the denominator

2.3 Work with quadratic functions and their graphs

The discriminant of a quadratic function, including the conditions for real and repeated roots Completing the square

Solution of quadratic equations, including solving quadratic equations in a function of the unknown

2.4 Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation

2.5 Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically, including inequalities with brackets and fractions

Express solutions through correct use of 'and' and 'or', or through set notation Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically

2.6 Manipulate polynomials algebraically, including expanding brackets, collecting like terms and factorisation and simple algebraic division; use of the factor theorem

2.7 Understand and use graphs of functions; sketch curves defined by simple equations including polynomials, $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ (including their vertical and horizontal asymptotes)

Interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations

2.9 Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs:

$y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A4 Collecting like terms and factorising

N8 Surds

A19 Solving linear simultaneous equations

A18 Solving quadratic equations (by factorising and completing the square)

A22 Working with inequalities

Solving quadratic inequalities

A12 Functional notation and shapes of standard graphs (e.g. parabola, cubic, reciprocal)

N7 Rules of indices

KEYWORDS

Expression, function, constant, variable, term, unknown, coefficient, index, linear, identity, simultaneous, elimination, substitution, factorise, completing the square, intersection, change the subject, cross-multiply, power, exponent, base, rational, irrational, reciprocal, root, standard form, surd, rationalise, exact, manipulate, sketch, plot, quadratic, maximum, minimum, turning point, transformation, translation, polynomial, discriminant, real roots, repeated roots, factor theorem, quotient, intercepts, inequality, asymptote .

1a. Algebraic expressions: basic algebraic manipulation, indices and surds (2.1) (2.2)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to perform essential algebraic manipulations, such as expanding brackets, collecting like terms, factorising etc;
- understand and be able to use the laws of indices for all rational exponents;
- be able to use and manipulate surds, including rationalising the denominator.

TEACHING POINTS

Recap the skills taught at GCSE Higher Tier (9-1).

Emphasise that in many cases, only a fraction or surd can express the exact answer, so it is important to be able to calculate with surds.

Ensure students understand that $\sqrt{a} + \sqrt{b}$ is *not* equal to $\sqrt{a+b}$ and that they know that $a^{\frac{m}{n}}$ is equivalent to $\sqrt[n]{a^m}$ and that a^{-m} is equivalent to $\frac{1}{a^m}$.

Most students understand the skills needed to complete these calculations but make basic errors with arithmetic leading to incorrect solutions.

Questions involving squares, for example $(2\sqrt{3})^2$, will need practice.

Students should be exposed to lots of simplifying questions involving fractions as this is where most marks are lost in exams.

Recap the difference of two squares $(x+y)(x-y)$ and link this to $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right) = x - y$, explaining the choice of term to rationalise the denominator.

Provide students with plenty of practice and ensure that they check their answers.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Include examples which involve calculating areas of shapes with side lengths expressed as surds. Exact solutions for Pythagoras questions is another place where surds occur naturally.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors include: misinterpreting $\left(a\sqrt{b}\right)^2$ as $(a+\sqrt{b})^2$; evaluating $(\sqrt{2})^2$ as 4 instead of 2; slips when multiplying out brackets; basic arithmetic errors; and leaving surds in the denominator rather than fully simplifying fractions. Two examples of errors with indices are, writing $\frac{1}{3x}$ as $3x^{-1}$and writing $\frac{4}{\sqrt{x}}$ as $4x^{\frac{1}{2}}$; these have significant implications later in the course (e.g. differentiation).

Many of these errors can be avoided if students carefully check their work and have plenty of practice.

NOTES

Make use of matching activities (e.g. Tarsia puzzles)

1b. Quadratic functions: factorising, solving, graphs and discriminants (2.3)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve a quadratic equation by factorising;
- be able to work with quadratic functions and their graphs;
- know and be able to use the discriminant of a quadratic function, including the conditions for real and repeated roots;
- be able to complete the square. e.g. $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$;
- be able to solve quadratic equations, including in a function of the unknown.

TEACHING POINTS

Lots of practice is needed as these algebraic skills are fundamental to all subsequent work. Students must become fluent, and continue to develop thinking skills such as choosing an appropriate method, and interpreting the language in a question. Emphasise correct setting out and notation.

Students will need lots of practice with negative coefficients for x squared and be reminded to always use brackets if using a calculator. e.g. $(–2)^2$.

Include manipulation of surds when using the formula for solving quadratic equations. [Link with previous sub-unit.]

Where examples are in a real-life contexts, students should check that solutions are appropriate and be aware that a negative solution may not be appropriate in some situations.

Students must be made aware that this sub-unit is about finding the links between completing the square and factorised forms of a quadratic and the effect this has on the graph. Use graph drawing packages to see the effect of changing the value of the '+ c' and link this with the roots and hence the discriminant.

Start by drawing $y = x^2$ and add different x terms followed by different constants in a systematic way. Then move on to expressions where the coefficient of x^2 is not 1.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Links can be made with Unit 3a – Proof:

Proof by deduction: e.g. complete the square to prove that $n^2 – 6n + 10$ is positive for all values of n.

Disproof by counter-example: show that the statement "$n^2 – n + 1$ is a prime number for all values of n" is untrue.

The path of an object thrown can be modelled using quadratic graphs. Various questions can be posed about the path:

- When is the object at a certain height?
- What is the maximum height?
- Will it clear a wall of a certain height, a certain distance away?

Areas of shapes where the side lengths are given as algebraic expressions.

Proof of the quadratic formula.

Working backwards, e.g. find a quadratic equation whose roots are $\frac{-5\pm\sqrt{17}}{4}$

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When completing the square, odd coefficients of x can cause difficulties. Students do not always relate finding the minimum point and line of symmetry to completing the square. Students should be provided with plenty of practice on completing the square with a wide range of quadratic forms.

Notation and layout can also be a problem; students must remember to show all the necessary working out at every stage of a calculation, particularly on 'show that' questions.

Examiners often refer to poor use of the quadratic formula. In some cases the formula is used without quoting it first and there are errors in substitution. In particular, the use of $-b \pm \frac{\sqrt{b^2-4ac}}{2a}$ so that the division does not extend under the "$-b$", is relatively common. Another common mistake is to think that the denominator is always 2. Also, students sometimes include x's in their expressions for the discriminant. Such methods are likely to lose a significant number of marks.

NOTES

Encourage the use of graphing packages or graphing Apps (e.g. Desmos or Autograph), so students can graph as they go along and 'picture' their solutions. You can link the discriminant with complex numbers if appropriate for students also studying Further Maths.

1c. Equations: quadratic/linear simultaneous (2.4)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve linear simultaneous equations using elimination and substitution;
- be able to use substitution to solve simultaneous equations where one equation is linear and the other quadratic.

TEACHING POINTS

Simultaneous equations are important both in future pure topics but also for applied maths. Students will need to be confident solving simultaneous equations including those with non-integer coefficients of either or both variables.

The quadratic may involve powers of 2 in one unknown or in both unknowns, e.g. Solve $y = 2x + 3$, $y = x^2 - 4x + 8$ or $2x - 3y = 6$, $x^2 - y^2 + 3x = 50$.

Emphasise that simultaneous equations lead to a pair or pairs of solutions, and that both variables need to be found.

Make sure students practise examples of worded problems where the equations need to be set up.

Students should be encouraged to check their answers using substitution.

Sketches can be used to check the number of solutions and whether they will be positive or negative. This will be reviewed and expanded upon as part of the curve sketching topic.

Use graphing packages or graphing Apps (e.g. Desmos or Autograph), so students can visualise their solutions e.g. straight lines crossing an ellipse or a circle.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Simultaneous equations in contexts, such as costs of items given total cost, can be used. Students must be aware of the context and ensure that the solutions they give are appropriate to that context.

Simultaneous equations will be drawn on heavily in curve sketching and coordinate geometry.

Investigate when simultaneous equations cannot be solved or only give rise to one solution rather than two.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Mistakes are often due to signs errors or algebraic slips which result in incorrect coordinates. Students should be encouraged to check their working and final answers, and if the answer seems unlikely to go back and look for errors in their working. Examiners often notice that it is the more successful candidates who check their solutions.

Students should remember to find the values of both variables as stopping after finding one is a common cause of lost marks in exam situations. Students who do remember to find the values of the second variable must take care that they substitute into a correct equation or a correctly rearranged equations.

1d. Inequalities: linear and quadratic (including graphical solutions) (2.5)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve linear and quadratic inequalities;
- know how to express solutions through correct use of ‘and’ and ‘or’ or through set notation;
- be able to interpret linear and quadratic inequalities graphically;
- be able to represent linear and quadratic inequalities graphically.

TEACHING POINTS

Provide students with plenty of practice at expressing solutions in different forms using the correct notation. Students must be able to express solutions using ‘and’ and ‘or’ appropriately, or by using set notation. So, for example:

$x < a$ **or** $x > b$ is equivalent to $\{x: x < a\} \cup \{x: x > b\}$

and $\{x: c < x\} \cap \{x: x < d\}$ is equivalent to $x > c$ **and** $x < d$.

Inequalities may contain brackets and fractions, but these will be reducible to linear or quadratic inequalities. For example, $\frac{a}{x} < b$ becomes $ax < bx^2$.

Students’ attention should be drawn to the effect of multiplying or dividing by a negative value, this must also be taken into consideration when multiplying or dividing by an unknown constant.

Sketches are the most commonly used method for identifying the correct regions for quadratic inequalities, though other methods may be used. Whatever their method, students should be encouraged to make clear how they obtained their answer.

Students will need to be confident interpreting and sketching both linear and quadratic graphs in order to use them in the context of inequalities.

Make sure that students are also able to interpret combined inequalities. For example, solving

$ax + b > cx + d$

$px^2 + qx + r \geq 0$

$px^2 + qx + r < ax + b$

and interpreting the third inequality as the range of x for which the curve $y = px^2 + qx + r$ is below the line with equation $y = ax + b$.

When representing inequalities graphically, shading and correctly using the conventions of dotted and solid lines is required. Students using graphical calculators or computer graphing software will need to ensure they understand any differences between the conventions required and those used by their graphical calculator.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Financial or material constraints within business contexts can provide situations for using inequalities in modelling. For those doing further maths this will link to linear programming.

Inequalities can be linked to length, area and volume where side lengths are given as algebraic expressions and a maximum or minimum is given.

Following on from using a quadratic graph to model the path of an object being thrown, inequalities could be used to find the time for which the object is above a certain height.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may make mistakes when multiplying or dividing inequalities by negative numbers.

In exam questions, some students stop when they have worked out the critical values rather than going on to identify the appropriate regions. Sketches are often helpful at this stage for working out the required region.

It is quite common, when asked to solve an inequality such as $2x^2 - 17x + 36 < 0$ to see an incorrect solution such as $2x^2 - 17x + 36 < 0 \Rightarrow (2x - 9)(x - 4) < 0 \Rightarrow x < \frac{9}{2}, x < 4$.

1e. Graphs: cubic, quartic and reciprocal (2.7)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and use graphs of functions;
- be able to sketch curves defined by simple equations including polynomials;
- be able to use intersection points of graphs to solve equations.

TEACHING POINTS

Students should be familiar with the general shape of cubic curves from GCSE (9-1) Mathematics, so a good starting point is asking students to identify key features and draw sketches of the general shape of a positive or negative cubic. Equations can then be given from which to sketch curves.

Quartic equations will be new to students and they may benefit from initially either plotting graphs by hand or using a graphical calculator or graphing software to look at the shape of the curve.

Cubic and quartic equations given at this point should either already be factorised or be easily simplified (e.g. $y = x^3 + 4x^2 + 3x$) as students will not yet have encountered algebraic division.

The coordinates of all intersections with the axes will need to be found. Where equations are already factorised, students will need to find where they intercept the axes. Repeated roots will need to be explicitly covered as this can cause confusion.

Students should also be able to find an equation when given a sketch on which all intersections with the axes are given. To do this they will need to be confident multiplying out multiple brackets.

Reciprocal graphs in the form $y = \frac{a}{x}$ are covered at GCSE but those in the form $y = \frac{a}{x^2}$ will be new. When sketching reciprocal graphs such as $y = \frac{a}{x}$ and $= \frac{a}{x^2}$, the asymptotes will be parallel to the axes.

Intersecting points of graphs can be used to solve equations, a curve and a line and two curves should be covered. When finding points of intersection students should be encouraged to check that their answers are sensible in relation to the sketch.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students should be able to justify the number of solutions to simultaneous equations using the intersections of two curves.

Students can be given sketches of curves or photographs of curved objects (e.g. roller coasters, bridges, etc.) and asked to suggest possible equations that could have been used to generate each sketch.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When sketching cubic graphs, most students are able to gain marks by knowing the basic shape and sketching it passing through the origin. Recognising whether the cubic is positive or negative sometimes causes more difficulty. Students sometimes fail to recognise the significance of a square factor in the factorised form of a polynomial.

When sketching graphs, marks can easily be lost by not labelling all the key points or labelling them incorrectly e.g. (0, 6) instead of (6, 0).

1f. Transformations: transforming graphs (2.9), f(x) notation (2.8)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$;
- be able to sketch the result of a simple transformation given the graph of any function $y = \mathrm{f}(x)$.

TEACHING POINTS

Transformations to be covered are: $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$ and $y = \mathrm{f}(ax)$.

Students should be able to apply one of these transformations to any of the functions listed and sketch the resulting graph:

quadratics, cubics, quartics, reciprocals, $y = \frac{a}{x^2}$, $\sin x$, $\cos x$, $\tan x$, e^x and a^x.

Students will need to be able to transform points and asymptotes both when sketching a curve and to give either the new point or the equation of the line.

Given a curve or an equation that has been transformed students should be able to state the transformation that has been used.

Links can be made with sketching specific curves. Students should be able to sketch curves like $y = (x - 3)^2 + 2$ and $y = \frac{2}{x-3} + 2$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Examples can be used in which the graph is transformed by an unknown constant and students encouraged to think about the effects this will have.

The use of graphing packages or graphing Apps (e.g. Desmos or Autograph) can be invaluable here.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

One of the most common errors is translating the curve in the wrong direction for $\mathrm{f}(x + a)$ or $\mathrm{f}(x) + a$. Students sometimes also apply the wrong scale factor when sketching $\mathrm{f}(ax)$.

Other errors involve algebraic mistakes and incomplete sketches, or sketches without key values marked.

Students should be encouraged to check any answers they have calculated against their sketches to check they make sense

NOTES

Dynamic geometry packages can be used to help students investigate and visualise the effect of transformations.

UNIT 2: Coordinate geometry in the (x, y) plane

Return to overview

SPECIFICATION REFERENCES

2.7 Understand and use proportional relationships and their graphs

3.1 Understand and use the equation of a straight line, including the forms $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$

Gradient conditions for two straight lines to be parallel or perpendicular

Be able to use straight line models in a variety of contexts

3.2 Understand and use the coordinate geometry of the circle including using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$

Completing the square to find the centre and radius of a circle

Use of the following properties:

- the angle in a semicircle is a right angle
- the perpendicular from the centre to a chord bisects the chord
- the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point

PRIOR KNOWLEDGE

Algebraic manipulation covered so far

- Simultaneous equations
- Completing the square

GCSE (9-1) in Mathematics at Higher Tier

A9 Equation of a line

Parallel and perpendicular lines

G20 Pythagoras

A14 Conversion graphs

R10 Calculating the proportionality constant k

G10 Circle theorems

KEYWORDS

Equation, bisect, centre, chord, circle, circumcircle, coefficient, constant, diameter, gradient, hypotenuse, intercept, isosceles, linear, midpoint, parallel, perpendicular, proportion, Pythagoras, radius, right angle, segment, semicircle, simultaneous, tangent.

2a. Straight-line graphs, parallel/perpendicular, length and area problems (3.1) (2.7)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and use the equation of a straight line;
- know and be able to apply the gradient conditions for two straight lines to be parallel or perpendicular;
- be able to find lengths and areas using equations of straight lines;
- be able to use straight-line graphs in modelling.

TEACHING POINTS

Students should be encouraged to draw sketches when answering questions or, if a diagram is given, annotate the diagram.

Equations can be given or asked for in the forms $y = mx + c$ and $ax + by + c = 0$ where a, b and c are integers. Students will need to be familiar with both forms, so questions should be asked where different forms are given or required in the answer. Given either form, students should be able to find the intercepts with the axes and the gradient. The x-intercept often causes students more difficulty, so will need more practice, but is useful for sketches and questions involving area or perimeter.

Students should be able to find the equation of a line given the gradient and a point, either the formula $y - y_1 = m(x - x_1)$ can be used or the values substituted into $y = mx + c$. To find the equation of a line from two points the gradient can be found and then one of the previous two methods used or the formula $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$ can be used. If this formula is used, care needs to be taken to ensure that the y-values are substituted into the correct places and that negative signs are taken into account. It should be emphasised that in the majority of cases, the form $y - y_1 = m(x - x_1)$ is far more efficient and less prone to errors than other methods.

The gradient conditions for parallel and perpendicular lines may be remembered from GCSE (9-1), but are still worth revising. They need to be well understood as they are used further when dealing with circles and in differentiation. Students should be able to identify whether lines are parallel, perpendicular or neither and find the equation of a parallel or perpendicular line when given a point on the line.

The length of a line segment is found by using Pythagoras' theorem, which can be written as the formula $d = \sqrt{(x - x_1)^2 + (y_2 - y_1)^2}$. This can be linked to proof with students being encouraged to show how to go from Pythagoras to the formula. Answers to length and distance questions are likely to be given in surd form, giving further practice in simplifying surds. Students should be encouraged to give answers in exact form unless specified otherwise.

Make shapes using lines and the axes; students can then be asked to find the area or perimeter of composite shapes. Answers should be given in exact form to practise combining and simplifying surds.

Real-life situations such as conversions can be modelled using straight-line graphs, this is likely to be familiar from GCSE (9-1) Mathematics.

Students should also be familiar with finding the relationship between two variables and expressing this using the proportion symbol $\propto$ or using an equation involving a constant (k). This can be extended to straight-line graphs through the origin with a gradient of k. Students should be able to calculate and interpret the gradient.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

To help students see how much information is given in the equation of a line, a good activity is to give an equation and ask students to find everything they know about that line, e.g. the intercepts, a point on the line, the gradient, a sketch, a parallel line, etc.

Students can be given sketches and asked to suggest equations that would/would not work.

Modelling with straight-line graphs gives the opportunity to collect data that can then be plotted and a line of best fit used to find an equation. It might also be possible to compare the data to a theoretical model. Students should be encouraged to consider strengths and limitations of modelling.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

In exams, students should be encouraged to quote formulae before using them. This allows method marks to be awarded even if arithmetical slips are made or incorrect values substituted.

Questions may specify a particular form for an answer (for example integer coefficients). Emphasise to students the importance of following these instructions carefully so as not to lose marks.

Students should be encouraged to draw diagrams while working on solutions as this often results in fewer mistakes and can act as a sense check for answers. At the same time, where diagrams are given in questions, students should be aware that these are not to be relied upon and 'spotting' answers by looking at a diagram without providing evidence to support this will not gain full marks. However, candidates should be encouraged to use any diagrams provided to help them answer the question.

The usual sorts of algebraic and numerical slips cause marks to be lost and students should be encouraged to carefully check their working. A common error is to incorrectly calculate the gradient of a straight line when it is given in the form $ax + by + c = 0$, so students should be encouraged to practice this technique.

NOTES

Dynamic geometry programs can be used to make changes and observe the effect, helping students to discover and visualise the effect of changing equations.

2b. Circles: equation of a circle, geometric problems on a grid (3.2)	**Teaching time** 7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the midpoint of a line segment;
- understand and use the equation of a circle;
- be able to find points of intersection between a circle and a line;
- know and be able to use the properties of chords and tangents.

TEACHING POINTS

Drawing sketches or annotating given diagrams will help students to understand the question in many cases and so should be encouraged.

Students should be able to find the midpoint given two points from GCSE (9-1) Mathematics. This can be built upon to find the coordinate of a point given the midpoint and one of the end points. The midpoint can be used to find the perpendicular bisector, recapping the work from straight-line graphs.

The equation of the circle $(x-a)^2+(y-b)^2=r^2$ can be derived from Pythagoras' theorem, giving students the opportunity to look at proof.

Students should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa. Students should be familiar with the equations $x^2+y^2+2fx+2gy+c=0$ and $(x-a)^2+(y-b)^2=r^2$. 'Complete the square' method should be used to factorise the equation into the more useful form. Students will need practice within this context to ensure that they are confident with the algebraic manipulation needed, in particular mistakes are often made with the signs and forgetting the constant term.

Circle theorems from GCSE (9-1) Mathematics can be used in questions so a quick recap could be useful and then they should be incorporated into questions. Examples of this include: finding the equation of the circumcircle of a triangle with given vertices; or finding the equation of a tangent using the perpendicular property of tangent and radius.

Simultaneous equations can be used to find the points of intersection between a circle and a straight line. Students can also be asked to show that a line and circle do not intersect, for which the discriminant can be used. Finding intersections with the axes should also be covered.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The conditions in which a circle and a line intersect can be investigated, with students justifying which will and will not intersect.

Investigate finding the equation of a circle given 3 points on its circumference.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Most errors when completing the square to find the equation of a circle involve the constant term. Students may forget to subtract it or perhaps add it instead. Having found the equation, when giving the coordinates of the centre students must take care to get the signs the right way round as marks are easily lost by getting this wrong.

When substituting into equations to find the intersections with axes, students sometimes substitute for the wrong variable, for example substituting $y = 0$ when trying to find the intersection with the y-axis. Another error is substituting the entire bracket $(x - a)$ for 0 rather than just x.

When finding the equation of a tangent to a point on the circle, typical errors are: finding the gradient of the radius; finding a line parallel to the radius; and finding a line through the centre of the circle.

UNIT 3: Further algebra

Return to overview

SPECIFICATION REFERENCES

2.6 Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem

1.1 Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including: proof by deduction, proof by exhaustion, disproof by counter-example

4.1 Understand and use the binomial expansion of $(a + bx)^n$ for positive integer n; the notations $n!$ and ${}_nC_r$; link to binomial probabilities

PRIOR KNOWLEDGE

Algebraic manipulation covered so far

- Factorising quadratics
- notation

GCSE (9-1) in Mathematics at Higher Tier

A4 Expanding brackets

A2 Substitution

A6 Proof

KEYWORDS

Binomial, coefficient, probability, proof, assumptions, deduction, exhaustion, disproof, counter-example, polynomials, factorisation, quadratic, cubic, quartic, conjecture, prediction, rational number, implies, necessary, sufficient, converse, fully factorise, factor, expand, therefore, conclusion.

3a. Algebraic division, factor theorem and proof (2.6) (1.1)

Teaching time
8 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use algebraic division;
- know and be able to apply the factor theorem;
- be able to fully factorise a cubic expression;
- understand and be able to use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion;
- be able to use methods of proof, including proof by deduction, proof by exhaustion and disproof by counter-example.

TEACHING POINTS

When using algebraic division, only division by ($ax + b$) or ($ax – b$) will be required.

Different methods for algebraic division should be considered depending on students' prior experience and preferred ways of working. Whichever method is used, clear working out should be shown.

Equations in which the coefficient of x or x^2 is 0 for example $x^3 + 3x^2 - 4$ or $2x^3 + 5x - 20$ will need additional explanation and practice.

Students should know that if f(x) = 0 when $x = a$, then ($x – a$) is a factor of f(x). Questions in the form ($ax + b$) should be covered.

Where a negative is being substituted into the equation the distinction between $(-2)^2$ and -2^2 will be important especially when students are using a calculator as examiners often comment on the fact that students will sometimes evaluate $(-2)^2$ as -4.

Factor theorem can be used to find an unknown constant. For example: Find a given that $(x - 2)$ is a factor of $x^3 + ax^2 - 4x + 6$. Two conditions can also be given in order to form simultaneous equations to solve.

When fully factorising a cubic, emphasis should be placed on choosing appropriate values. The final answer may need to be written as a factorised cubic or, alternatively, as the solutions to an equation which can then be used to sketch the curve. Students sometimes use the roots of a polynomial equation to help them factorise but this method must be used with care. Questions sometimes use the word 'hence' and so students must be careful which method they chose in these cases.

This is an excellent opportunity to review curve sketching by asking students to give a sketch following factorisation.

Students should be familiar with basic proofs from GCSE (9-1) Mathematics this knowledge can be built upon to look at the different types of proof. Students will need to understand how to set out each type of proof; the correct conventions in language and layout should be encouraged.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The factor theorem can be introduced through investigation by substituting different values and checking against division to look for patterns.

Proof gives the opportunity to review previous concepts in a different way for example coordinate geometry. Proof will also be included in later topics.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The majority of errors seen in exam questions are not due to misunderstanding the method, but instead arithmetic and algebraic mistakes. For example, incorrect simplification of terms – especially those involving fractions; mistakes with negative numbers; and writing expressions rather than equations.

Students should be aware that long division is not always the best or quickest method to use and sometimes results in some complicated algebra.

When using the factor theoremf, stress the importance of checking the value that is substituted; a common error is to use, for example, f(1) rather than f(–1).

You should also emphasise the importance of fully factorising expressions, as a fairly significant number of students stop when they have reached one linear factor and a quadratic factor.

3b. The binomial expansion (4.1)

Teaching time
7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the binomial expansion of $(a + bx)^n$ for positive integer n;
- be able to find an unknown coefficient of a binomial expansion.

TEACHING POINTS

Students should initially be introduced to Pascal's triangle, which can be used to expand simple brackets.

Students will need to be familiar with factorials and the ${}_nC_r$ notation.

Introduce the formal binomial expansion in the same way as the formula booklet and discuss the various terms to ensure all students understand.

Setting out work clearly and logically will be invaluable in helping students to achieve the final answer and also to spot mistakes if necessary.

Where there is a coefficient of x (other than 1) students will need to be reminded that the power applies to the whole term, not just the x, and that answers must be simplified appropriately. Negative and fractional coefficients will also need practice.

The limitations of the binomial expansion should be discussed.

Students should practice finding the coefficient of a single term, they should also be able to deal with setting up simple algebraic equations to find unknown constants.

Use of the binomial expansion can be linked to basic probability and approximations.

[Links can also be made with the statistics work in A level Mathematics.]

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can be encouraged to discover the link between Pascal's triangle and the expansion of simple brackets.

Students could look at find the general term of a particular expansion.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks are most commonly lost in exam questions because of errors in expanding terms. For example not including the coefficient when calculating, say, $(ax)^2$; not simplifying terms fully; sign errors; and omitting brackets. Good notation will help to avoid many of these mistakes.

When writing expansions which involve unknown constants, some students fail to also include the x's in their expansion.

When using their expansions to work out the value of a constant, a significant number of students do not understand that the coefficient does not include the x or x^2 part and so are often unable to form an equation in the unknown alone.

Questions often go on to ask students to use their binomial expansion to evaluate a number raised to a power. For example, evaluating $(1.025)^8$ by substituting $x = 0.025$ into an expansion for $(1 + x)^8$. Students should be advised that simply using their calculator to evaluate $(1.025)^8$ will gain no marks as it is not answering the question.

NOTES

Be aware of an alternative notation such as $\binom{n}{r}$ and ${}^{n}C_{r}$.

UNIT 4: Trigonometry

Return to overview

SPECIFICATION REFERENCES

5.1 Understand and use the definitions of sine, cosine and tangent for all arguments; the sine and cosine rules; the area of a triangle in the form $\frac{1}{2}ab\sin C$

5.3 Understand and use the sine, cosine and tangent functions; their graphs, symmetries and periodicity

5.5 Understand and use $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Understand and use $\sin^2\theta + \cos^2\theta = 1$

5.7 Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle

PRIOR KNOWLEDGE

Algebra covered so far

- Basic algebraic manipulation
- Quadratics
- Graph transformations

GCSE (9-1) in Mathematics at Higher Tier

G20 Pythagoras' Theorem

Trigonometry in right-angled triangles

G22 The sine rule

The cosine rule

G23 The area of a triangle

G15 Bearings

KEYWORDS

Sine, cosine, tangent, interval, period, amplitude, function, inverse, angle of elevation, angle of depression, bearing, degree, identity, special angles, unit circle, symmetry, hypotenuse, opposite, adjacent, intercept.

4a. Trigonometric ratios and graphs (5.1) (5.3)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the definitions of sine, cosine and tangent for all arguments;
- understand and be able to use the sine and cosine rules;
- understand and be able to use the area of a triangle in the form $\frac{1}{2}ab\sin C$;
- understand and be able to use the sine, cosine and tangent functions; their graphs, symmetries and periodicity.

TEACHING POINTS

Students should be shown the x and y coordinates of points on the unit circle can be used to give cosine and sine respectively.

Use of trigonometric ratios will have been covered at GCSE (9-1) Mathematics; questions should now be focused more on multi-step problems and questions set in context.

When using the sine rule the ambiguous case should be covered.

Links to proof can be made, for example proving the area of a triangle.

Students should be encouraged to write down any formulae they will be using before substituting in the numbers.

Students should be able to solve questions in various contexts; these could include coordinate geometry or real-life situations. Questions may involve bearings, which may not be well remembered from GCSE so should be reviewed. Students should be encouraged to check that their answers are realistic as this check can show up errors.

When completing multi-step questions emphasise to students that they should show all working out and use the answer function on their calculators to avoid rounding errors. It can be a useful teaching point to divide the class asking one side to round all answers and the other to keep values stored in their calculator to show how this affects the final answer.

The unit circle can again be used to show how the trigonometric graphs are formed. Characteristics such as the period and amplitude should be discussed. Knowledge of graphs of curves with equations such as $y = \sin x$, $y = \cos(x + 30)$, $y = \tan 2x$ is expected so this is a good opportunity to recap transformations.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Use of the graphs can be linked to modelling situations such as yearly temperatures, wave lengths and tidal patterns.

Proof of the sine and cosine rules.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students occasionally assume that triangles given in exam questions are right-angled and so use right-angled trigonometric ratios rather than the sine and cosine rules.

A frequently seen error in these questions is students using the cosine rule to calculate an incorrect angle, sometimes despite having drawn a correctly labelled diagram. This indicates a lack of understanding of how the labelling of edges and angles on a diagram relates to the application of the cosine rule formula.

4b. Trigonometric identities and equations (5.5) (5.7)	**Teaching time** 10 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve trigonometric equations within a given interval
- understand and be able to use $\tan\theta = \frac{\cos\theta}{\sin\theta}$
- Understand and use $\sin^2\theta + \cos^2\theta = 1$

TEACHING POINTS

When solving trigonometric equations, finding multiple values within a range can initially be illustrated using the graphs of the functions. The decision can then be made whether to move on to using CAST diagrams or continue using graphs. Whichever method is used students will need plenty of practice in identifying all values within the limits correctly.

Intervals with negative solutions as well as positive solutions should be used.

Students should be able to solve equations such as $\sin(x + 70°) = 0.5$ for $0 < x < 360°$; $3 + 5\cos 2x = 1$ for $–180° < x < 180°$; and $6\cos 2x + \sin x - 5 = 0$ for $0 < x < 360°$, giving their answers in degrees.

Students should be comfortable factorising quadratic trigonometric equations and finding all possible solutions. It should be noted that in some cases only one of the factorisations will give solutions but in most case there will be two sets of solutions. Situations where one answer is equal to zero can cause some confusion with students then not looking for further solutions. This sort of example should be covered in class. For example, the equation, $\sin\theta(3\sin\theta + 1) = 0$ will often be simplified to just $3\sin\theta + 1 = 0$, resulting in the loss of solutions to the original equation.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Following from the previous section, if graphs are used to model situations then the equations can be used to find values at given points.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors include: not finding values in the given range; finding extra, incorrect, solutions; not going on to find the values of x and instead leaving the values for, say $2x$ or $x + 30$; algebraic slips when rearranging the equation; and not giving answers to the correct degree of accuracy. The loss of accuracy in the final answers to trigonometric equations is common and often results in the unnecessary loss of marks.

Sketches of the trigonometric functions are often helpful to check all solutions have been found.

UNIT 5: Vectors (2D)

Return to overview

SPECIFICATION REFERENCES

10.1 Use vectors in two dimensions

10.2 Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form

10.3 Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations

10.4 Understand and use position vectors; calculate the distance between two points represented by position vectors

10.5 Use vectors to solve problems in pure mathematics and in context, (including forces)

PRIOR KNOWLEDGE

Covered so far

- Surds

GCSE (9-1) in Mathematics at Higher Tier

G24 Vectors

KEYWORDS

Vector, scalar, magnitude, direction, component, parallel, perpendicular, modulus, dimension, ratio, collinear, scalar product, position vectors.

5a. Definitions, magnitude/direction, addition and scalar multiplication (10.1) (10.2) (10.3)	Teaching time 7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use vectors in two dimensions;
- be able to calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form;
- be able to add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.

TEACHING POINTS

Students need to be familiar with column vectors and with the use of **i** and **j** vectors in two dimentions.

Students should be able to find a unit vector in the direction of $\boldsymbol{a}$, and be familiar with the notation $|\boldsymbol{a}|$.

The triangle and parallelogram laws of addition should be known and students should be able to use them. Students should understand that vectors are commutative.

Where answers are given in surds they should be simplified if possible.

When performing operations on vectors this should also be understood geometrically, diagrams will be helpful here. Students should be able to use given diagrams but also draw their own in order to assist with questions.

Students should understand and be able to use the conditions for parallel vectors.

Use the classroom floor as a 2-dimensional grid to help students visualise vectors. Use the position of students in the room to illustrate concepts.

Consider vectors in the real world, e.g. ask students to think of everyday phenomena that have a magnitude and direction e.g. forces, velocities, displacements.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can prove vectors are parallel to demonstrate their reasoning skill.

Given particular vectors, students can investigate places they can or cannot reach, for example the knights problem on a chessboard.

Consider an aircraft landing in a cross-wind – what direction does it need to fly?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students sometimes make mistakes when manipulating vectors in **i** and **j** form and should be encouraged to use column vectors when possible.

5b. Position vectors, distance between two points, geometric problems (10.4) (10.5)	Teaching time 7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use position vectors;
- be able to calculate the distance between two points represented by position vectors;
- be able to use vectors to solve problems in pure mathematics and in context, (including forces).

TEACHING POINTS

Students should know and be able to use $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Students should be able to calculate the distance between two points (x_1, y_1) and (x_2, y_2) using the formula $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

Use the ratio theorem to find the position vector of a point *C* dividing *AB* in a given ratio.

Use familiar shapes to illustrate the difference between 2 vectors and vector addition, e.g. parallelogram, rectangle.

When solving problems using vectors only pure contexts are covered.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Finding position vector of the fourth corner of a shape (e.g. parallelogram) *ABCD* with three given position vectors for the corners *A*, *B* and *C*.

Use regular polygons to find vectors connecting different vertices and to illustrate the ratio theorem.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiners comment that students understand the simple basics of vectors but are unable to deal with the complexity of ratios. Students should be given plenty of practice in identifying points that divide line segments in a particular ratio both externally and internally.

UNIT 6: Differentiation

Return to overview

SPECIFICATION REFERENCES

7.1 Understand and use the derivative of f(x) as the gradient of the tangent to the graph of y = f(x) at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change

Sketching the gradient function for a given curve

Second derivatives

Differentiation from first principles for small positive integer powers of x

7.2 Differentiate x^n, for rational values of n, and related constant multiples, sums and differences

7.3 Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points

Identify where functions are increasing or decreasing

PRIOR KNOWLEDGE

Covered so far

- Solving quadratics
- Coordinate geometry
- Proof
- Function notation
- Indices

GCSE (9-1) in Mathematics at Higher Tier

N8 Fractions

G16 Area of 2D shapes

Volume and surface area of 3D shapes

A5 Rearranging equations

KEYWORDS

Differentiation, derivative, first principles, rate of change, rational, constant, tangent, normal, increasing, decreasing, stationary point, maximum, minimum, integer, calculus, function, parallel, perpendicular.

6a. Definition, differentiating polynomials, second derivatives (7.1) (7.2)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the derivative of f(x) as the gradient of the tangent to the graph of y = f(x) at a general point (x, y);
- understand the gradient of the tangent as a limit and its interpretation as a rate of change;
- be able to sketch the gradient function for a given curve;
- be able to find second derivatives;
- understand differentiation from first principles for small positive integer powers of x;
- be able to differentiate x^n, for rational values of n, and related constant multiples, sums and differences.

TEACHING POINTS

Students should know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the rate of change of y with respect to x.

Knowledge of the chain rule is not required.

The notation f′(x) may be used for the first derivative and f″(x) may be used for the second order derivative.

Students should be able to identify maximum and minimum points as points where the gradient is zero.

Cover the use of the second derivative to establish the nature of a turning point.

Students should be able to sketch the gradient function f′(x) for a given curve y = f(x), using given axes and scale. This could involve speed and acceleration for example.

Students should know how to differentiate from first principles. Students should be able to use, for n =2 and n = 3, the gradient expression $\lim_{h\to 0}\left(\frac{(x+h)^n-x^n}{h}\right)$. The alternative notations $h \to 0$ rather than $\delta x \to 0$ are acceptable.

Students will need to be confident in algebraic manipulation of functions to ensure that they are in a suitable format for differentiation. For example, students will be expected to be able to differentiate expressions such as $(2x + 5)(x - 1)$ and $\frac{x^2+3x-5}{4x^{\frac{1}{2}}}$, for $x > 0$. Mistakes are easily made with negative and/or fractional indices so there should be plenty of practice with this.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Maxima and minima problems set in the context of a practical problem, e.g. minimising the materials required to make a container of a particular shape. The open box problem – simple cases and the general case.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Algebraic manipulation, particularly where surds are involved, can cause problems for students. For example, when multiplying out brackets and faced with $-4\sqrt{x} \times -4\sqrt{x}$ common incorrect answers are $-4\sqrt{x}$, $\pm 16\sqrt{x}$ and $\pm 16x^{\frac{1}{4}}$. Similarly, when dividing by $\sqrt{x}$, some students think that $\frac{x}{\sqrt{x}}$ = 1.

6b. Gradients, tangents, normals, maxima and minima (7.3)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points;
- be able to identify where functions are increasing or decreasing.

TEACHING POINTS

Students should be able to use differentiation to find equations of tangents and normals at specific points on a curve. This reviews and extends the earlier work on coordinate geometry.

Maxima, minima and stationary points can be used in curve sketching. Problems may be set in the context of a practical problem. This could bring in area and volume from GCSE (9-1) Mathematics as well as using trigonometry.

Students will need plenty of practice at setting up equations from a given context, in some cases this may include showing that it can be written in a particular form. Where students are given the answer to work towards they must be aware that they need to work forwards showing all steps clearly rather than starting with the answer and working backwards.

Students need to know how to identify when functions are increasing or decreasing. For example, given that $f'(x) = x^2 - 2 + \frac{1}{x^2}$, , prove that f($x$) is an increasing function.

Use graph plotting software that allows the derivative to be plotted so that students can see the relationship between a function and its derivative graphically.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Differentiation can be linked to many real-world applications, there can be discussion with students about contexts and the validity of solutions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may have difficult differentiating fractional terms such as $\frac{8}{x}$ if they are unable to rewrite this as $8x^{-1}$ before differentiating.

When working out the equations of tangents and normal, some students mix the gradients and equations up and end up substituting in the wrong place.

Questions involving finding a maximum or minimum point do require the use of calculus and attempts using trial and improvement will receive no marks.

When finding a stationary point, some students use inequalities as their condition rather than equating their derivative to zero. Another error is to differentiate twice and solve $f''(x) = 0$.

When applying differentiation in context, students should be ensure they give full answers and not just a partial solution. For example if asked to find the volume of a box they must not stop after finding the side length.

UNIT 7: Integration

Return to overview

SPECIFICATION REFERENCES

8.1 Know and use the Fundamental Theorem of Calculus

8.2 Integrate x^n (excluding $n = -1$), and related sums, differences and constant multiples

8.3 Evaluate definite integrals; use a definite integral to find the area under a curve

PRIOR KNOWLEDGE

Covered so far

- Algebraic manipulation
- Differentiation

KEYWORDS

Calculus, differentiate, integrate, reverse, indefinite, definite, constant, evaluate, intersection.

7a. Definition as opposite of differentiation, indefinite integrals of x^n (8.1) (8.2)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know and be able to use the Fundamental Theorem of Calculus;
- be able to integrate x^n (excluding $n = -1$), and related sums, differences and constant multiples.

TEACHING POINTS

Integration can be introduced as the reverse process of differentiation. Students need to know that for indefinite integrals a constant of integration is required.

Similarly to differentiation, students should be confident with algebraic manipulation. For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$ and $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected. Introduce students to the integral sign; this can be useful in setting work out clearly on these sorts of questions and will be used later in definite integration.

Given $f'(x)$ and a point on the curve, students should be able to find an equation of the curve in the form $y = f(x)$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students should be able to explain the need for the $+ c$ in indefinite integration.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students sometimes have difficulty when integrating expressions involving negative indices. Forgetting to add $+ c$ when working out indefinite integrals is also a very common mistake.

7b. Definite integrals and areas under curves (8.3)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to evaluate definite integrals;
- be able to use a definite integral to find the area under a curve.

TEACHING POINTS

It is important that students show their working out clearly as mistakes are easily made when putting values into a calculator. Students should also be encouraged to check their answers. Calculators that perform numerical integration can be used as a check, but a full method will be needed.

Students will be expected to understand the implication of a negative answer from indefinite integration.

Links can be made with curve sketching in questions where students need to find the points of intersection with the x-axis for a curve in order to find the limits of integration.

Areas can be made up of a combination of a curve and a line so further links can be made to coordinate geometry.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Discuss the implication of a negative answer to encourage students reasoning skills.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Lack of algebraic fluency can cause problems for some students, particularly when negative/fractional indices are involved or when a negative number is raised to a power. Arithmetic slips are also a common cause of lost marks, often when negative numbers are substituted and subtracted after integration.

Students are generally more successful if they expand any brackets before attempting to integrate the function.

UNIT 8: Exponentials and logarithms
Exponential functions and natural logarithms
(6.1) (6.2) (6.3) (6.4) (6.5) (6.6) (6.7)

Teaching time
12 hours

Return to overview

SPECIFICATION REFERENCES

6.1 Know and use the function a^x and its graph, where a is positive
Know and use the function e^x and its graph

6.2 Know that the gradient of e^{kx} is equal to $k\mathrm{e}^{kx}$ and hence understand why the exponential model is suitable in many applications

6.3 Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$
Know and use the function $\ln x$ and its graph
Know and use $\ln x$ as the inverse function of e^x

6.4 Understand and use the laws of logarithms:

$$\log_a x + \log_a y = \log_a (xy)$$
$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$
$$k \log_a x = \log_a x^k \text{(including, for example, } k = -1 \text{ and } = -\tfrac{1}{2}\text{)}$$

6.5 Solve equations of the form $a^x = b$

6.6 Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y

6.7 Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models

PRIOR KNOWLEDGE

Covered so far

- Indices

GCSE (9-1) in Mathematics at Higher Tier

R16 Compound interest

KEYWORDS

Exponential, exponent, power, logarithm, base, initial, rate of change, compound interest

OBJECTIVES

By the end of the sub-unit, students should:

- know and be able to use the function a^x and its graph, where a is positive;
- know and be able to use the function e^x and its graph;
- know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications;
- know and be able to use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$;
- know and be able to use the function $\ln x$ and its graph;
- know and be able to use $\ln x$ as the inverse function of e^x ;
- understand and use the laws of logarithms:

  $\log_a x + \log_a y = \log_a (xy)$

  $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

  $k\log_a x = \log_a x^k$(including, for example, $k = –1$ and $k = -\frac{1}{2}$)
- be able to solve equations of the form $a^x = b$;
- be able to use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y;
- understand and be able to use exponential growth and decay in modelling, giving consideration to limitations and refinements of exponential models.

TEACHING POINTS

When sketching the graph of a^xstudents should understand the difference in shape between $a < 1$ and $a > 1$.

Explain to students that e^x is a special case of a^x. Graphs of the function e^x should include those in the form $y = e^{ax+b} + c$.

Students should realise that when the rate of change is proportional to the y-value, an exponential model should be used.

An ability to solve equations of the form $e^{ax+b} = p$ and $\ln (ax + b) = q$ is expected.

Students can use the laws of indices to prove the laws of logarithms and show that $\log_a a = 1$.

In solving equations students may use the change of base formula. Solving equations questions may be in the form $2^{3x-1} = 3$.

Students should be able to plot $\log y$ against $\log x$ and obtain a straight line where the intercept is $\log_a$ and the gradient is n and plot $\log y$ against x to obtain a straight line where the intercept is $\log k$ and the gradient is $\log b$. There should be discussion about why this is an appropriate model and why it is only an estimate.

Contexts for modelling should could include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth. Students should be familiar with terms such as initial, meaning when $t = 0$. They may need to explore the behaviour for large values of t or to consider whether the range of values predicted is appropriate. Consideration of a second improved model may be required.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can look at different models for population growth using the exponential function.

Use graphing software to investigate varying the parameters of a population model.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Errors seen in exam questions where students have to sketch exponential curves include: stopping the curve at $x = 0$; getting the wrong y-intercept; and believing the curve levels off to $y = 1$ for $x < 0$.

When using laws of logs to answer proof or 'show that' questions, students must show all the steps clearly and not have jumps in their working out.

Year 1: AS Mathematics applied content
Section A: Statistics

Unit		Title	Estimated hours
1		**Statistical sampling**	
	a	Introduction to sampling terminology; Advantages and disadvantages of sampling	**1**
	b	Understand and use sampling techniques; Compare sampling techniques in context	**2**
2		**Data presentation and interpretation**	
	a	Calculation and interpretation of measures of location; Calculation and interpretation of measures of variation; Understand and use coding	**4**
	b	Interpret diagrams for single-variable data; Interpret scatter diagrams and regression lines; Recognise and interpret outliers; Draw simple conclusions from statistical problems	**8**
3		**Probability:** Mutually exclusive events; Independent events	**3**
4		**Statistical distributions:** Use discrete distributions to model real-world situations; Identify the discrete uniform distribution; Calculate probabilities using the binomial distribution (calculator use expected)	**5**
5		**Statistical hypothesis testing**	
	a	Language of hypothesis testing; Significance levels	**2**
	b	Carry out hypothesis tests involving the binomial distribution	**5**
			30 hours

UNIT 1: Statistical sampling

Return to overview

SPECIFICATION REFERENCES

1.1 Understand and use the terms ‘population’ and ‘sample’
Use samples to make informal inferences about the population
Understand and use sampling techniques, including simple random sampling and opportunity sampling.
Select or critique sampling techniques in the context of solving a statistical problem, including understanding that different samples can lead to different conclusions about the population

PRIOR KNOWLEDGE

GCSE (9–1) in Mathematics at Higher Tier

S1 Infer properties of populations or distributions from a sample, while knowing the limitations of sampling

S5 Apply statistics to describe a population

KEYWORDS

Population, census, sample, sampling unit, sampling frame, simple random sampling, stratified, systematic, quota, opportunity (convenience) sampling.

1a. Introduction to sampling terminology; Advantages and disadvantages of sampling (1.1)	Teaching time 1 hour

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the terms ‘population’ and ‘sample’;
- know how to use samples to make informal inferences about the population;
- be able to describe advantages and disadvantages of sampling compared to census.

TEACHING POINTS

This section is a great opportunity to introduce the large data set to look at a population of data and discuss reasons for sampling from it.

Students will be expected to be able to comment on the advantages and disadvantages associated with a census and a sample.

Discuss in context the meanings of populations and samples. Look at data from populations and samples, initially using data from the sample to make inferences about the population before then checking the data for the population.

Discuss the advantages and disadvantages of sampling making sure to include time, cost etc.

Ensure students are given the opportunity, and are able, to give full and thorough answers within the context of the question.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

The biggest opportunity here is introducing students to the large data set and starting to get them familiar with the data included in it.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some students confuse sample sizes and population sizes, but the recurring problem is not giving answers in context. Candidates need to be clear about the difference between sample sizes and population sizes.

1b. Understand and use sampling techniques; Compare sampling techniques in context (1.1)	Teaching time 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use sampling techniques;
- be able to describe advantages and disadvantages of sampling techniques;
- be able to select or critique sampling techniques in the context of solving a statistical problem;
- understand that different samples can lead to different conclusions about the population.

TEACHING POINTS

Students will also be expected to be familiar with different types of sampling including simple random, stratified, systematic, quota and opportunity (convenience) sampling.

Students will gain a more thorough understanding of the types of sampling if the advantages and disadvantages alongside the method used for each type are understood. They will then be more able to select an appropriate technique for a given statistical problem and be able to critique a technique which has been used.

Give students the opportunity to use the techniques they learn about on the large data set.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Again, this is a perfect opportunity to use the large data set and discuss how different samples from the same data set could lead to different conclusions.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students need to be able to describe the sampling techniques clearly and will lose marks if they are not sufficiently precise.

As always, answers must be given using the context of the question and not simply be quoted from text books in a general form.

UNIT 2: Data presentation and interpretation

Return to overview

SPECIFICATION REFERENCES

2.1 Interpret diagrams for single-variable data, including understanding that area in a histogram represents frequency

Connect to probability distributions

2.2 Interpret scatter diagrams and regression lines for bivariate data, including recognition of scatter diagrams which include distinct sections of the population (calculations involving regression lines are excluded)

Understand informal interpretation of correlation

Understand that correlation does not imply causation

2.3 Interpret measures of central tendency and variation, extending to standard deviation
Be able to calculate standard deviation, including from summary statistics

2.4 Recognise and interpret possible outliers in data sets and statistical diagrams
Select or critique data presentation techniques in the context of a statistical problem
Be able to clean data, including dealing with missing data, errors and outliers

PRIOR KNOWLEDGE

GCSE (9–1) in Mathematics at Higher Tier

S2 Interpret and construct tables, charts and diagrams, including frequency tables, bar charts, pie charts and pictograms for categorical data, vertical line charts for ungrouped discrete numerical data and know their appropriate use

S3 Construct and interpret diagrams for grouped discrete data and continuous data, i.e. histograms with equal and unequal class intervals and cumulative frequency graphs, and know their appropriate use

S4 Interpret, analyse and compare the distributions of data sets from univariate empirical distributions through appropriate measures of central tendency (median, mean, mode and modal class) and spread (range, including consideration of outliers), quartiles and inter-quartile range

S6 Use and interpret scatter graphs of bivariate data; recognise correlation and know that it does not indicate causation; draw estimated lines of best fit; make predictions; interpolate and extrapolate apparent trends while knowing the dangers of so doing

KEYWORDS

Histogram, box plot, probability density function, cumulative distribution function, continuous random variable, scatter diagram, linear regression, explanatory (independent) variables, response (dependent) variables interpolation, extrapolation, product moment correlation coefficient (PMCC), mean, median, mode, variance, standard deviation, range, interquartile range, interpercentile range, outlier, skewness, symmetrical, positive skew, negative skew.

2a. Calculation and interpretation of measures of location; Calculation and interpretation of measures of variation; Understand and use coding (2.3) (2.4)

Teaching Time
4 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate measures of location, mean, median and mode;
- be able to calculate measures of variation, standard deviation, variance, range and interpercentile range;
- be able to interpret and draw inferences from summary statistics.

TEACHING POINTS

The calculation of the mean, median and mode should be recapped from GCSE however the focus now is on students using calculators to do the calculations. Check understanding of the terminology and teach calculator methods.

Students require an understanding of measures of variation too and should be able to use their calculators to calculate the variance and standard deviation. They should be able to use the statistic $S_{xx} = \sum(x-\bar{x})^2 = \sum x^2 - \frac{\left(\sum x\right)^2}{n}$. Students are expected to use standard deviation $= \sqrt{\frac{S_{xx}}{n}}$ but equivalents including spreadsheet formula ($s = \sqrt{\frac{S_{xx}}{n-1}}$) will be accepted.

The data may be discrete or continuous, grouped or ungrouped, and students need to be able to interpret these summary statistics clearly and be able to make inferences from them. Significance tests will not be expected.

Coding for both mean and standard deviation needs to be covered. Be clear that students need to be able to uncode both mean and standard deviation. Emphasise that the standard deviation is unaffected by the addition or subtraction of constants.

Students are expected to be able to use linear interpolation to calculate percentiles from grouped data.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

There is opportunity for further use of the large data set here. Summary statistics of elements from the data set can be calculated and then used to compare and interpret for both location and variation statistics.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When calculating the mean, of grouped data some student may divide by the number of groups rather than the number of items of data, they may also use class widths in the calculation rather than the mid-points.

When finding the standard deviation, the most common error is forgetting to take the square root (perhaps because they are not clear about the difference between variance and standard deviation). Some students waste time by ignoring given values and recalculating Σfx and Σfx^2.

Difficulties with coding are due to a lack of understanding about how coding affects the mean and standard deviation, and poor algebraic skills. Students sometimes substitute for the wrong variable, fail to solve equations correctly or get the order of operations the wrong way around.

Students should be reminded that they must be precise in their use of language and use the correct terms such as 'median'. 'range' or 'inter-quartile range' rather than the more general 'average' and 'spread'. Students should also remember to use accurate values throughout calculations to avoid losing marks due to premature rounding.

NOTES

Students are expected to know the different notation for population summary statistics $\left(\mu, \sigma^2, \sigma\right)$ and sample summary statistics $\left(\bar{x}, s^2, s\right)$.

2b. Interpret diagrams for single-variable data; Interpret scatter diagrams and regression lines; Recognise and interpret outliers; Draw simple conclusions from statistical problems (2.1) (2.2) (2.4)	Teaching time 8 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know how to interpret diagrams for single variable data;
- know how to interpret scatter diagrams and regression lines for bivariate data;
- recognise the explanatory and response variables;
- be able to make predictions using the regression line and understand its limitations;
- understand informal interpretation of correlation;
- understand that correlation does not imply causation;
- recognise and interpret possible outliers in data sets and statistical diagrams;
- be able to select or critique data presentation techniques in the context of a statistical problem;
- be able to clean data, including dealing with missing data, errors and outliers.

TEACHING POINTS

Students should be familiar with and be able to interpret histograms, frequency polygons, box and whisker plots and cumulative frequency diagrams. These should have been covered at GCSE but it is worth a recap for consistency of methods. Also cover calculating summary statistics from diagrams, including the mean and standard deviation from a histogram.

For bivariate data students should understand the terms explanatory and response variables and know where each is placed on the axes of a scatter diagram. This is particularly important as variables other than y and x could be used.

Students are not expected to know, calculate or understand the regression line formula. Students will need to understand the use of interpolation when using a regression line equation to make predictions within the range of values of the explanatory variable and they need to understand the dangers of extrapolation (predictions outside the range), again variables other than y and x could be used.

Students will be expected to describe the correlation on a scatter diagram in terms of positive, negative or no correlation and strong or weak but no calculations need to be made. Values from calculations will not be given for interpretation.

Outliers will need to be identified and interpreted from data sets and statistical diagrams. Any rules to be used will be given in the question, for example $Q_1 - 1.5 \times IQR$, $Q_3 + 1.5 \times IQR$.

Students will be expected to select an appropriate diagram or critique the choice of one which is used. They should also be able to clean data by identifying possible outliers (box plots and scatter diagrams). They may also be asked to fill in missing data using a regression line.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Again all of the diagrams and techniques used in this unit could be modelled using data from the large data set.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Knowing how to interpret statistics students have calculated is sometimes found challenging, and often discriminates between students in exam questions. Full and clear reasons for interpretations and decisions need to be given for marks to be awarded.

Many students have difficulties calculating the sizes of bars in histograms, as commented on by one examiner: ‘Most were able to state the correct width of the bar but few used frequency densities correctly to find the height, some finding the frequency density of but then calculating $\frac{1}{3} \times 2.5$ rather than $2.5 \div \frac{1}{3}$. Some identified that 1.5 cm^2 represented 10 customers but were then unable to use this correctly to find the height … some students had an incorrect class width because they did not realize that the lower class boundary was 70 not 69.5.’

UNIT 3: Probability Mutually exclusive events; Independent events (3.1)	Teaching time 3 hours

Return to overview

SPECIFICATION REFERENCES

3.1 Understand and use mutually exclusive and independent events when calculating probabilities Link to discrete and continuous distributions

PRIOR KNOWLEDGE

GCSE (9–1) in Mathematics at Higher Tier

P1 Record, describe and analyse the frequency of outcomes of probability experiments using tables and frequency trees

P2 Apply ideas of randomness, fairness and equally likely events to calculate expected outcomes of multiple future experiments

P3 Relate relative expected frequencies to theoretical probability, using appropriate language and the 0–1 probability scale

P4 Apply the property that the probabilities of an exhaustive set of outcomes sum to one; apply the property that the probabilities of an exhaustive set of mutually exclusive events sum to one

P6 Enumerate sets and combinations of sets systematically, using tables, grids

P7 Construct theoretical possibility spaces for single and combined experiments with equally likely outcomes and use these to calculate theoretical probabilities

P9 Tree diagrams and Venn diagrams

KEYWORDS

Sample space, exclusive event, complementary event, discrete random variable, continuous random variable, mathematical modelling, independent, mutually exclusive, Venn diagram, tree diagram.

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use mutually exclusive and independent events when calculating probabilities;
- be able to make links to discrete and continuous distributions.

TEACHING POINTS

Tree and Venn diagrams should have been covered at GCSE but will need to be recapped as one way of looking at probabilities.

The focus at this level is on independent and mutually exclusive events in probability calculations. Students should be confident in the definitions of both independent and mutually exclusive events and how to use their properties to solve real-life probability problems.

Cover showing independence but be aware that the use of set notation is not required at AS level. At this level this is done by showing the product of the probabilities of two events gives the probability of both events occurring together. Understanding of conditional probability is not expected at AS level.

Students do not need to be aware of probability density functions however they should understand that probability is represented by the area under a curve in a continuous distribution. This could be mentioned here and comparisons drawn by using the binomial model as a bar chart in the next unit.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Include questions of the type where A and B are independent which use Venn diagrams and informal use of the addition rule but where both $P(A)$ and $P(A \cap B)$ for example are unknown; the solution relies on a knowledge of independence. (Set notation not required)

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may confuse ‘independent’ and ‘mutually exclusive’.

Using a diagram almost always helps students to answer probability questions. When drawing a Venn diagram, students should remember to include a box defining the universal set.

UNIT 4: Statistical Distributions **Use discrete distributions to model real-world situations; Identify the discrete uniform distribution; Calculate probabilities using the binomial distribution (calculator use expected) (4.1)**	**Teaching time** 5 hours

Return to overview

SPECIFICATION REFERENCES

4.1 Understand and use simple, discrete probability distributions (calculation of mean and variance of discrete random variables is excluded), including the binomial distribution, as a model; calculate probabilities using the binomial distribution

PRIOR KNOWLEDGE

An understanding of probability from the previous unit and the awareness that the area under a curve will be looked at again in this unit.

GCSE (9–1) in Mathematics at Higher Tier

N1 Order positive and negative integers, decimals and fractions; use the symbols $=$, $\neq$, $<$, $>$, $\leq$, and $\geq$

KEYWORDS

Binomial, probability, discrete distribution, discrete random variable, uniform, cumulative probabilities.

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use simple, discrete probability distributions, including the binomial distribution;
- be able to identify the discrete uniform distribution;
- be able to calculate probabilities using the binomial distribution.

TEACHING POINTS

Students will be expected to model real-world situations by using simple discrete probability distributions. They should know and be able to recognise a discrete uniform distribution; look at equally likely outcomes such as numbers on a dice.

The only specific distribution students are expected to use as well as understand is the binomial distribution. Students will be expected to comment critically on how appropriate a given probability model may be for a situation.

The notation $X \sim \mathrm{B}(n, p)$ may be used, so you should ensure students are familiar with this from the outset. Make sure the properties of the binomial are clear for all students, so that they know a fixed number of trials is needed, there are only two possible outcomes per trial and the outcome of each trial is independent.

Once the binomial distribution has been introduced link back to thinking about probability being the area under a curve. Use a bar chart for discrete binomial distributions and show how this would smooth into a curve if it were a continuous distribution. Another teaching point for this concept of area could come from considering the discrete uniform distribution as bars of equal width; it looks like a rectangle, like the continuous uniform distribution.

Students need to calculate probabilities using the binomial distribution for both individual and cumulative probabilities. Calculator use is expected for all of this, so time needs to be spent making sure students are competent in the use of these calculator functions.

The bar chart model mentioned earlier helps students distinguish between for example $\mathrm{P}(X < 2)$ and $\mathrm{P}(X \leq 2)$, also to understand $\mathrm{P}(X \geq 6) = 1 - \mathrm{P}(X \leq 5)$. Explain this is due to the binomial being a discrete distribution. This is essential when manipulating before using the calculator to find probabilities. Encourage students to shade the bars required to help with this understanding.

Emphasise the importance of reading questions carefully. The probability of success can be worded negatively in the question for example ‘the probability of people failing their driving test first time is 0.6’.

Students are not expected to be able to calculate the mean and variance of discrete random variables.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Look at a wide variety of real-world scenarios and model using a number of different distributions to ensure students are fluent in their comments on the appropriateness of a particular distribution.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common difficulty is with manipulating inequalities: ‘A significant number of students were unable to cope with the expression $\mathrm{P}(5 \le X < 11)$. There were students who translated this expression into the more convenient form $\mathrm{P}(5 \le X \le 10)$ and then in turn transformed this into an equivalent form that can be applied to the table of cumulative probabilities: $\mathrm{P}(X \le 10) - \mathrm{P}(X \le 4)$. However, there were also many instances of incorrect versions such as: $\mathrm{P}(X < 11) - \mathrm{P}(X \ge 5)$, $\mathrm{P}(X \le 10) + \mathrm{P}(X \ge 5)$, $\mathrm{P}(X \le 10) - (1 - \mathrm{P}(X \ge 5))$ and $\mathrm{P}(X \le 11)$ – either $\mathrm{P}(X \le 5)$ or $\mathrm{P}(X \le 4)$.’

In a similar vein, students have a tendency to write, for example, $\mathrm{P}(X > 2)$ as $1 - \mathrm{P}(X \le 1)$ instead of $1 - \mathrm{P}(X \le 2)$.

NOTES

It would be good for understanding to see what the random variable looks like in pdf form and table form although this is not explicitly in the specification.

UNIT 5: Statistical hypothesis testing

Return to overview

SPECIFICATION REFERENCES

5.1 Understand and apply the language of statistical hypothesis testing, developed through a binomial model: null hypothesis, alternative hypothesis, significance level, test statistic, 1-tail test, 2-tail test, critical value, critical region, acceptance region, p-value

5.2 Conduct a statistical hypothesis test for the proportion in the binomial distribution and interpret the results in context

Understand that a sample is being used to make an inference about the population and appreciate that the significance level is the probability of incorrectly rejecting the null hypothesis

PRIOR KNOWLEDGE

An understanding of how to calculate binomial probabilities and using samples from populations from previous units.

KEYWORDS

Hypotheses, significance level, one-tailed test, two-tailed test, test statistic, null hypothesis, alternative hypothesis, critical value, critical region, acceptance region, p-value, binomial model, accept, reject, sample, inference.

5a. Language of hypothesis testing; Significance levels (5.1)	Teaching time 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to apply the language of statistical hypothesis testing, developed through a binomial model.

TEACHING POINTS

The concept of a hypothesis could be introduced initially by posing some hypotheses yourself. You may wish to make reference to the large data set again and say for example 'the daily maximum temperature was higher in Hurn than Heathrow in May 1987'.

Following this introduce the null and alternative hypotheses and their respective notation H_0 and H_1. Discuss how to move from statements like the one above to using the language of the binomial distribution in terms of looking at p, the probability of success.

The focus of this sub-unit is the language used in terms of hypothesis testing, but a scenario must be set. You may wish to use an example like 'the number of 6s thrown in 50 throws of a dice', students could carry out this experiment and you could use their results to form a variety of tests which would cover all of the terminology without actually carrying out the tests. Make sure you save these examples to be tested in the next sub-unit.

All of the terms from the keywords section should be thoroughly discussed and understood before attempting to carry out a hypothesis test.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Use a wide variety of scenarios from the real world and invite students to offer their own scenarios; discuss their suitability for hypothesis testing.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Emphasise the importance of stating hypotheses clearly using the correct notation.

Similarly, correct notation is important when describing the critical region: 'There were still a few students using incorrect notation for critical regions: $P(X \leq 1)$, for example, is not a critical region: it is a probability.'

NOTES

The expected value of the binomial distribution being np needs to be appreciated for a two-tailed test.

5b. Carry out hypothesis tests involving the binomial distribution (5.2)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to conduct a statistical hypothesis test for the proportion in the binomial distribution and interpret the results in context;
- understand that a sample is being used to make an inference about the population;
- appreciate that the significance level is the probability of incorrectly rejecting the null hypothesis.

TEACHING POINTS

Once all the terminology that has been discussed in the previous sub-unit is fully understood, you can go back to the examples you used and conduct the hypothesis tests. Carry out the tests both by finding the critical value to compare with your test statistic and by finding the probability (p-value) of the test statistic and comparing it with the critical region. Ensure students are competent with both methods. Make sure hypotheses are always written clearly in terms of ρ, the probability of success.

Spend time making sure that students can write clear and concise conclusions in the given context of the questions.

When using a sample of data, ensure students understand, what it infers about the population itself.

Type I errors are not part of the specification, but it is important that students understand what the significance level of a test actually means. Discuss carefully that rejecting the null hypothesis may actually be incorrect and the significance level is the probability of this. Also cover ‘the actual significance level of a test’ with students.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Again, use a wide variety of scenarios from the real world and make sure all conclusions are written in these contexts.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

The most common error in these sorts of questions include not writing a clear conclusion in the context of the question. Students either omit the context or sometimes fail to give any conclusion to their calculations.

Year 1: AS Mathematics applied content
Section B – Mechanics

Unit		Title	Estimated hours
6		**Quantities and units in mechanics**	
	a	Introduction to mathematical modelling and standard S.I. units of length, time and mass	1
	b	Definitions of force, velocity, speed, acceleration and weight and displacement; Vector and scalar quantities	2
7		**Kinematics 1 (constant acceleration)**	
	a	Graphical representation of velocity, acceleration and displacement	4
	b	Motion in a straight line under constant acceleration; *suvat* formulae for constant acceleration; Vertical motion under gravity	6
8		**Forces & Newton's laws**	
	a	Newton's first law, force diagrams, equilibrium, introduction to $\mathbf{i}$, $\mathbf{j}$ system	4
	b	Newton's second law, '$F = ma$', connected particles (no resolving forces or use of $F = \mu R$); Newton's third law: equilibrium, problems involving smooth pulleys	6
9		**Kinematics 2 (variable acceleration)**	
	a	Variable force; Calculus to determine rates of change for kinematics	4
	b	Use of integration for kinematics problems i.e. $r=\int v\,dt$, $v=\int a\,dt$	3
			30 hours

UNIT 6: Quantities and units in mechanics

Return to overview

SPECIFICATION REFERENCES

6.1 Understand and use fundamental quantities and units in the S.I. system: length, time, mass.
Understand and use derived quantities and units: velocity, acceleration, force, weight

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

R1 Change freely between related standard units (e.g. time, length, area, volume/capacity, mass) and compound units (e.g. speed, rates of pay, prices, density, pressure) in numerical and algebraic contexts

R11 Use compound units such as speed, rates of pay, unit pricing, density and pressure

A14 Plot and interpret graphs (including reciprocal graphs and exponential graphs) and graphs of non-standard functions in real contexts to find approximate solutions to problems such as simple kinematic problems involving distance, speed and acceleration

A15 Calculate or estimate gradients of graphs and area under graphs (including quadratic and non-linear graphs), and interpret results in cases such as distance-time graphs, velocity-time graphs and graphs in financial contexts

KEYWORDS

Modelling, smooth, rough, light, inelastic, inextensible, particle, rigid body, mass, weight, rod, plane, lamina, length, distance (m), displacement (m), velocity (m s^{-1}), speed (m s^{-1}), acceleration (m s^{-2}), force (N), retardation (m s^{-2}), newtons (N), scalar, vector, direction, magnitude, (normal) reaction, friction, tension, thrust, compression

NOTES

There may not be a direct examination question on this topic. However, the modelling process and fluent knowledge of the S.I. units is a vital pre-requisite that underpins the rest of the mechanics course.

6a. Introduction to mathematical modelling and standard S.I units of length, time and mass (6.1)

Teaching time
1 hour

OBJECTIVES

By the end of the sub-unit, students should:

- understand the concept of a mathematical model, and be able to abstract from a real-world situation to a mathematical description (model);
- know the language used to describe simplifying assumptions;
- understand the particle model;
- be familiar with the basic terminology for mechanics;
- be familiar with commonly-made assumptions when using these models;
- be able to analyse the model appropriately, and interpret and communicate the implications of the analysis in terms of the situation being modelled;
- understand and use fundamental quantities and units in the S.I. system: length, time and mass;
- Understand that units behave in the same way as algebraic quantities, e.g. meters per second is $m/s = m \times 1/s = ms^{-1}$.

TEACHING POINTS

Begin by asking students ‘What is mechanics?’ Lead them to the idea that mechanics is a branch of applied mathematics that deals with motion and the forces producing motion.

Students need to be comfortable with the idea that mathematics is used to model real life and need to become familiar with the modelling cycle:

mechanics problem → create a mathematical model (using diagrams, general principles or formulae) → solve the model → refer back to the original problem → refine the model

[Link with the data-handling cycle]

It is important for students to get a ‘feel’ for mechanics at this early stage in order to support later work.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Examples of problems that may be solved in this way include:

- How far apart should the cameras be within an average speed zone?
- At what angle should you hold an umbrella to keep snow off you?

Some examples of simplifying assumptions for these problems include:

- treating the car as a particle
- motion is in a straight line
- snow falls vertically.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students can generally correctly state assumptions, but they need to make sure that any assumptions or statements about the model relate directly to the context they are considering. For example they could make the comment ‘the resistance will not be constant’ more specific by saying ‘resistance will increase as velocity increases’.

NOTES

The particle (point mass) model is introduced here, i.e. the body has no size but does have mass, so rotation is ignored and the forces all act at one point.
The language of simplifying assumptions (light, smooth, uniform, inextensible, thin, rigid etc) is mostly introduced in subsequent sections.

6b. Definitions of force, velocity, speed, acceleration, weight and displacement; Vector and scalar quantities (6.1)

Teaching time
1 hour

OBJECTIVES

By the end of the sub-unit, students should:

- understand and use derived quantities and units: velocity, acceleration, force, weight;
- know the difference between position, displacement and distance;
- know the difference between velocity and speed, and between acceleration and magnitude of acceleration;
- know the difference between mass and weight (including gravity);
- understand that there are different types of forces.

TEACHING POINTS

Revise GCSE (9-1) in Mathematics compound units for speed and acceleration and make sure that students are comfortable converting from one unit to another, e.g. from km h^{-1} into m s^{-1}.

Define the vector quantities displacement and velocity as the vector versions of distance and speed respectively.

Begin by walking across the room and explaining the difference between position (referred to a fixed origin), displacement (a vector measured from any position) and distance (a scalar quantity for the total movement). Then move onto discussing speed (the rate at which an object covers distance) and velocity (the rate of change of displacement or speed in a certain direction.

Mention the special acceleration (for a falling object) due to gravity. In this course, this value is assumed to be a constant g, usually 9.8 m s^{-2} though it does vary in the real world.

This could be a good opportunity to dispel common misconceptions around weight and mass. Make it clear that mass is the amount of 'stuff' something is made of, is a scalar and is fixed (in kg), whereas weight is a force of attraction between an object and the centre of the earth and can vary depending on gravity and is measured in newtons. Hence weight = mass × gravity (or $W = mg$).

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Show some basic force diagrams (as an introduction to Unit 8a) to illustrate different types of forces such as weight, reaction and tension (all in newtons).

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

As mentioned above, students may mix up mass and weight and their related units. Some struggle to use the correct vocabulary e.g. for velocity and displacement. It is important to be really clear when giving the definitions and to always use the correct vocabulary in discussions.

NOTES

Defining the units of acceleration as 'metres per second per second' helps explain the concept of rate of change of speed. Show that m/s/s is algebraically equivalent to ms^{-2}. It may also help to think about it in terms of 'how many metres per second of speed is the object gaining every second?'

UNIT 7: Kinematics 1 (constant acceleration)

Return to overview

SPECIFICATION REFERENCES

7.1 Understand and use the language of kinematics: position; displacement; distance travelled; velocity; speed; acceleration

7.2 Understand, use and interpret graphs in kinematics for motion in a straight line: displacement against time and interpretation of gradient; velocity against time and interpretation of gradient and area under the graph

7.3 Understand, use and derive the formulae for constant acceleration for motion in a straight line

8.3 Understand and use weight and motion in a straight line under gravity; gravitational acceleration, *g*, and its value in S.I. units to varying degrees of accuracy

PRIOR KNOWLEDGE

<u>GCSE (9-1) in Mathematics at Higher Tier</u>

R1 Change freely between related standard units (e.g. time, length, area, volume/capacity, mass) and compound units (e.g. speed, rates of pay, prices, density, pressure) in numerical and algebraic contexts

R11 Use compound units such as speed, rates of pay, unit pricing, density and pressure

A2 Substitute numerical values into formulae and expressions, including scientific formulae

A5 Understand and use standard mathematical formulae; rearrange formulae to change the subject

A14 Plot and interpret graphs (including reciprocal graphs and exponential graphs) and graphs of non-standard functions in real contexts to find approximate solutions to problems such as simple kinematic problems involving distance, speed and acceleration

A15 Calculate or estimate gradients of graphs and area under graphs (including quadratic and non-linear graphs), and interpret results in cases such as distance-time graphs, velocity-time graphs and graphs in financial contexts

A17 Solve linear equations in one unknown algebraically (including those with the unknown on both sides of the equation)

A18 Solve quadratic equations (including those that require rearrangement) algebraically by factorising, by completing the square and by using the quadratic formula

<u>AS Mathematics – Pure Mathematics content</u>

3.1 Gradient (See Unit 2a of the SoW)

KEYWORDS

Distance (m), displacement (m), speed (m s^{-1}), velocity (m s^{-1}), acceleration (m s^{-2}), retardation (m s^{-2}), deceleration (m s^{-2}), scalar, vector, 2D, linear, area, trapezium, gradient, equations of motion, gravity, constant, 9.8 m s^{-2}, vertical.

NOTES

The guidance on the specification document states that graphical solutions to problems may be required. This section assumes *constant* acceleration; hence the graphical approach involves linear line segments and the familiar equations of linear motion *suvat*, formulae for constant acceleration. (N.B. 'equation of motion' refers to $F = ma$, and is nothing to do with these formulae).

The guidance also states that derivation of constant acceleration formulae may use knowledge of sections 7.2 and/or 7.4 (Unit 9).

Kinematics 2 (Unit 9) analyses particles' motion under a variable force, hence a variable acceleration. The mathematical model for this requires calculus which is covered in AS Mathematics – Pure Mathematics content, see SoW Units 6 and 7.

The usual value for g in this course is 9.8 m s^{-2}, but some questions may specify a different value. Students may assume that g is constant, but should be aware that it is not a universal constant but depends on location.

7a. Graphical representation of velocity, acceleration and displacement (7.1) (7.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to draw and interpret kinematics graphs, knowing the significance (where appropriate) of their gradients and the areas underneath them.

TEACHING POINTS

Introduce this topic by making links to the GCSE (9-1) in Mathematics prior knowledge for distance–time (travel) and speed–time graphs. Kinematics is the analysis of a particle's motion without reference to the resultant force that caused that motion.

Stress that forces causing the motion of the body in this section are *constant*, therefore acceleration is constant and this results in a *straight line* travel speed-time or velocity-time graph.

Extend the ideas to displacement by considering a particle which moves in reverse direction back beyond the starting point.

For a velocity–time graph, consider the units for the area of a unit square 1 m s^{-1} by 1 s. The 's' cancels, leaving 'm', therefore the area represents the displacement.

Discuss and interpret graphs that model real situations. For example, the distance–time graph for a particle moving with constant speed, the velocity–time graph for a particle with constant acceleration.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Throw an object straight (vertically) up in the air. Time the flight and estimate the greatest height, to scale the graphs correctly, and keep for possible later use. Draw the displacement–time and velocity–time graphs (upward direction positive and initial velocity non-zero). If the object is caught at the same height at which it was thrown, what is the average velocity for the motion?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Many students can draw a velocity time graph with the correct shape, but do not always label the required speeds and times clearly on the axes. Students often tend to add a scale (for example 4, 8, 12, 16, …) unnecessarily, rather than just indicating the initial and final speeds.

Candidates are able to find distance travelled and the acceleration from velocity–time graphs and can find an average speed, but some struggle with the vocabulary of velocity and displacement.

NOTES

This unit can be linked with Unit 7b by drawing a general velocity–time graph for a particle with initial velocity (u), final velocity (v), taking time (t) moving under constant acceleration (a). The gradient of the line is $\frac{v-u}{t} = a$, which rearranges to $v = u + at$. Finding the area under the graph in 3 different ways will lead to 3 of the other *suvat* formulae but $v^2 = u^2 + 2as$ will have to be derived by eliminating t between two of them.

7b. Motion in a straight line under constant acceleration; *suvat* formulae for constant acceleration; Vertical motion under gravity (7.3) (8.3)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- recognise when it is appropriate to use the *suvat* formulae for constant acceleration;
- be able to solve kinematics problems using constant acceleration formulae;
- be able to solve problems involving vertical motion under gravity.

TEACHING POINTS

Make links back to Unit 7a and contrast the previous graphical approach with this algebraic approach. Note that there are five quantities, s, u, v, a and t (four vectors and one scalar) and each formula relates four of them hence there are five formulae. The formulae that must be derived and learnt are:

- $v = u + at$
- $s = \frac{(u+v)t}{2}$
- $s = ut + \frac{1}{2}at^2$
- $v^2 = u^2 + 2as$
- $s = vt - \frac{1}{2}at^2$

These formulae are only valid for *constant* acceleration in a straight line (and are referred to as the *suvat* formulae).

When solving problems, write down known variables and the variable(s) to be found – this should help to identify which one (or more, as some problems will involve simultaneous equations) of the *suvat* formulae to select. Emphasise to students the need to make sure units are compatible.

Model the good practice of drawing a diagram to illustrate the situation whenever possible, especially when considering vertical motion under gravity. This will encourage students to draw their own diagrams.

Mark the positive direction on the diagram and take acceleration due to gravity (g) to be 9.8 m s^{-2} unless directed otherwise. Students may assume that g is constant, but they should be aware that g is not a universal constant but depends on location.

If an object is thrown upwards and upwards is taken as being positive then $a = -9.8$ m s^{-2}. Explain that the velocity is zero at the greatest height and there is symmetry in the path (up and down to the same point) due to the fact that we model air resistance as being negligible.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

One of the more demanding problems is when two objects are released (or dropped) at different times, say 2 seconds apart, and students are asked to find the common position when one catches-up or passes the other. Students may find it difficult to select the times (values of t) to assign in the equations; they may need guiding towards t and $(t - 2)$ or t and $(t + 2)$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students are generally able to use *suvat* formulae in 2D to find unknown heights, velocities etc. However, students sometimes ignore the significance of a negative value for velocity, acceleration or displacement and don’t refer their answer back to the original problem. They need to recognise that $s = -3$ m means the object is 3 m *below* its starting point in the negative direction i.e. s is effectively a coordinate. This is where a diagram helps students understand the physics of the situation.

NOTES

End the section by looking forwards to Kinematics 2 (Unit 9) with a problem illustrating a variable acceleration e.g. $v = 2t^2 + 3t$. Explain that this will give a curved velocity–time graph so the *suvat* formulae will not work and instead we may need to find the gradient of the tangent and the area under the curve (Link back to GCSE (9-1) in Mathematics at Higher Tier or to calculus in AS Mathematics – Pure Mathematics content, see SoW Unit 7) You could also consider how to analyse motion in 2D (or 3D); this will be addressed during Kinematics 2 (Unit 9).

UNIT 8: Forces and Newton’s laws

Return to overview

SPECIFICATION REFERENCES

8.1 Understand the concept of a force; understand and use Newton’s first law.

8.2 Understand and use Newton’s second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2D (**i**, **j**) vectors).

8.4 Understand and use Newton’s third law; equilibrium of forces on a particle and motion in a straight line; application to problems involving smooth pulleys and connected particles.

PRIOR KNOWLEDGE

- Modelling and definitions/assumptions from the introduction in Unit 6

GCSE (9-1) in Mathematics at Higher Tier

A19 Solve two simultaneous equations in two variables (linear/linear or linear/quadratic) algebraically; find approximate solutions using a graph

AS Mathematics – Pure Mathematics content

10.1 – 10.5 Vectors in 2D (See SoW Unit 5)

KEYWORDS

Force, newtons, mass, weight, gravity, tension, thrust, compression, air resistance, reaction, driving force, braking force, resultant, force diagram, equilibrium, inextensible, light, negligible, particle, smooth, uniform, pulley, string, retardation, free particle.

NOTES

This section does *not* contain any resolving of forces into perpendicular components, nor does it consider the use of the coefficient of friction for frictional forces.

8a. Newton's first law, force diagrams, equilibrium, introduction to i, j system of vectors (8.1)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the concept of a force; understand and use Newton's first law.

TEACHING POINTS

Relate this topic back to the different types of forces defined in Unit 6b.

Newton said '*An object continues in state of rest or uniform motion unless acted on by an external force*.' Hence one can define a force as something which causes a body to accelerate. Explain to students that 'no force acting' means a body will either be stationary or be moving with constant velocity (i.e. acceleration = zero). This is why in outer space an object keeps moving at constant speed once pushed (there are no forces to speed it up, slow it down or stop it moving.)

So, an object at rest or constant velocity $\Rightarrow$ no resultant force; an object changing speed or direction $\Rightarrow$ resultant force. This will lead to Newton's second law in the next section.

Newton also stated '*When an object A exerts a force on another object B there is an equal and opposite reaction force of B on A.*' Explain that if a book is on a smooth, horizontal table, the forces acting on the book are the Weight, W (vertically down) and the normal reaction, R (always at 90° to the *surface* of contact). Assuming the table surface material is strong enough to hold the full weight of the book, the two forces balance each other and there is no resultant force. The book does not move, hence it is in equilibrium.

Ask questions such as: If the book has a mass of 5 kg, what is its weight? Therefore, what would the magnitude of the normal reaction be to guarantee equilibrium?

Draw different examples of force diagrams to illustrate: weight, reaction, tension (in strings), thrust (in rods), compression (in light rods, springs) etc.

To illustrate thrust, balance a book on a ruler.In which direction is the thrust force acting?

Introduce the **i** - **j** notation. The forces can be given in **i** - **j** form or as column vectors. Questions on equilibrium will be limited to perpendicular forces so the sum of the forces must be $0\mathbf{i} + 0\mathbf{j}$ for equilibrium.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

You could extend to rough inclined planes and show all the forces balancing to provide equilibrium. Resolving forces is not in the AS course (covered in A Level Mathematics – Mechanics content, see SoW Unit 5a).

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students are often good at drawing force diagrams, but common errors are omitting arrowheads, incorrectly labelling (e.g. 4 kg rather than 4*g*) and missing off the normal reaction.

Students can easily be confused by the vocabulary, e.g. mixing up 'resultant' and 'reaction'.

NOTES

Resolving forces is not in the AS course and equilibrium problems will not require forces to be resolved. Scenarios will be restricted to forces in two perpendicular directions, or simple cases of forces, given as 2D vectors.

Resolving forces and the concept of a friction force (which opposes relative motion) is covered in A level Mathematics – Mechanics section, see SoW Unit 5.

8b. Newton's second law, '$F = ma$', connected particles (no resolving forces or use of $F = \mu R$); Newton's third law: equilibrium, problems involving smooth pulleys (8.2) (8.4)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use Newton's second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2D (**i**, **j**) vectors.);
- understand and use Newton's third law; equilibrium of forces on a particle and motion in a straight line; application to problems involving smooth pulleys and connected particles.

TEACHING POINTS

Newton stated, '*Where there is a force, there is an acceleration (or deviation from uniform motion) and the force is proportional to the acceleration*'. Therefore $F \propto a$, and choosing the constant to suit the motion units gives $F = ma$. (Newton's second law). This is known as the 'equation of motion'.

Explain to students that if they sum all the effects of the forces acting, in a particular direction, this will be equal to the mass x the acceleration in that direction. This process is called resolving the forces in that direction e.g. resolving horizontally, or R(→) for short. It's usually best to resolve IN the direction of the acceleration and/or perpendicular to the direction of the acceleration.

When resolving always take the positive direction as the direction of the acceleration and put all the forces on one side of the equation and (mass x acceleration) on the other side.

When working on connected particles problems (such as trains or pulley systems) explain to students that they should consider the whole system as well as the separate parts. Applications to be covered are lift problems, car and caravan type questions and connected particles passing over a smooth pulley. Consider both pulley scenarios: a pulley with both stings hanging vertically; and a pulley at the end of a horizontal table.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

For the connected particle problems, discuss, the assumptions from Unit 6a, i.e. smooth pulley, inextensible string, same tension in the string. Extend the questions so that (for a pulley question) the particle moving down eventually hits the table and the string goes slack. This means the particle moving up continues as a 'free' particle so we now apply the equations of motion with $a = -9.8 \text{ m s}^{-2}$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Pulleys: In past exam questions, most students used an equation of motion for each particle with very few 'single equation' solutions. Students may also mistakenly take the acceleration to be equal to g rather than the value obtained in the question.

2 Vehicles: In exam questions of a car-and-trailer type, students may consider the car and trailer as a single system. Common errors when resolving are: to add a tension force (when there is no rope): to consider the weight; or to confuse the positive and negative directions.

NOTES

Starting with the ‘winner’ is most useful when the dynamics move into more complicated questions later in the course (e.g. inclined planes and resolving forces).

UNIT 9: Kinematics 2 (variable acceleration)

Return to overview

SPECIFICATION REFERENCES

7.4 Use calculus in kinematics for motion in a straight line.

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A11 Identify and interpret roots, intercepts, turning points of quadratic functions graphically; deduce roots algebraically and turning points by completing the square

A14 Plot and interpret graphs (including reciprocal graphs and exponential graphs) and graphs of non-standard functions in real contexts to find approximate solutions to problems such as simple kinematic problems involving distance, speed and acceleration

A15 Calculate or estimate gradients of graphs and area under graphs (including quadratic and non-linear graphs), and interpret results in cases such as distance-time graphs, velocity-time graphs and graphs in financial contexts

AS Mathematics – Pure Mathematics content

7, 8 Differentiation and integration of polynomials (See Units 6 and 7 of the SoW)

KEYWORDS

Distance, displacement, velocity, speed, constant acceleration, variable acceleration, retardation, deceleration, gradient, area, differentiate, integrate, rate of change, straight-line motion, with respect to time, constant of integration, initial conditions.

NOTES

All the functions in this section are functions of time, so the differentials and integrals are always with respect to time.

9a. Variable force; Calculus to determine rates of change for kinematics (differentiation) (7.4)	Teaching time 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use calculus (differentiation) in kinematics to model motion in a straight line for a particle moving with variable acceleration;
- understand that gradients of the relevant graphs link to rates of change;
- know how to find max and min velocities by considering zero gradients and understand how this links with the actual motion (i.e. acceleration = 0).

TEACHING POINTS

Start by stating that the *suvat* formulae from Unit 7 can only be used when acceleration is constant and the motion is in a straight line. This means the speed-time or velocity-time graphs are made up of straight lines.

Draw the graph of say, $v = 2t^2 + 2t + 1$ (for $t > 0$). This is part of a parabola where the gradient is increasing so as time passes the object is accelerating more quickly. As acceleration is not constant, the *suvat* formulae will not work for this model.

Make links (using AS Pure Mathematics calculus) to the rate of change of velocity explaining that $\frac{\mathrm{d}v}{\mathrm{d}t}$ = gradient = acceleration. This idea that the gradient of a velocity–time graph gives acceleration should be familiar from previous work in Unit 7 and also from GCSE (9-1) in Mathematics.

Summarise the situation by talking about, velocity as the rate of change of displacement and acceleration as the rate of change of velocity.

Express these statements in the notation of calculus: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ and $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$.

Students will also need to relate the fact that the gradient = 0 at the max or min point to this mathematical model i.e. if $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$, then acceleration = 0, so the particle must be at max or min velocity, as it cannot accelerate (or get any faster or slower) any more at this point in time.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

You could extend the calculus approach to relate double differentiation and signs of $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ to indicate if it is a min or max displacement.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students who draw sketches of the situation are often more successful in reaching the correct solution, so you should continue to encourage this wherever possible.

Students often ignore or don't recognise the difference between displacement and distance and so may end up discarding negative values without considering how they should be interpreted.

NOTES

The level of calculus will be consistent with the contents of AS Pure Mathematics.

The specification states the following:– $v = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$ using '$\boldsymbol{r}$' to represent displacement 's'.
r will become the vector notation of displacement when we later analyse 2D kinematics using the **i**, **j** system (A level Mathematics – Mechanics section, see SoW Unit 8).

9b. Use of integration for kinematics problems i.e. $\boldsymbol{r} = \int v \, \mathrm{d}t$, $v = \int a \, \mathrm{d}t$ (7.4)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use calculus (integration) in kinematics to model motion in a straight line for a particle moving under the action of a variable force;
- understand that the area under a graph is the integral, which leads to a physical quantity;
- know how to use initial conditions to calculate the constant of integration and refer back to the problem.

TEACHING POINTS

Return to the graph of $v = 2t^2 + 2t + 1$ (for $t > 0$) introduced at the start of Unit 9a.

From earlier work in Unit 7 and from GCSE (9-1) in Mathematics, students should know that the area under a velocity–time graph equals the displacement.

Remind students that, from their work for Pure Mathematics, the area under a curve can be found using integration. This means that the integral of the velocity expression (with respect to time) gives the displacement.

By linking integration with the reverse of differentiation, displacement and velocity can be found by integrating expressions for velocity and acceleration respectively:

$$\boldsymbol{r} = \int v \, \mathrm{d}t \text{ and } v = \int a \, \mathrm{d}t$$

(Again 's' can be used in place of '$\boldsymbol{r}$' for straight line motion in this section)

Move on to explain that the constant of integration, c needs to be found by referring back to the problem and using some (usually initial) information about the body. For example knowing that the particle starts from O at rest means that when $t = 0$ (initially), $s = 0$ (at O) and $v = 0$ (at rest). These values can be substituted to calculate c.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Students need to be able to know when to differentiate and/or integrate and how acceleration = 0 gives a maximum velocity so questions like the following are useful.

A particle moves so that it's motion is modelled by the following equation, $v = 6t(3 - t)$ m s^{-1}.

Find: **a** the times when it is at rest, **b** its maximum velocity, **c** an expression for its acceleration, **d** the total distance it travels between the times it is stationary.

Extension: Starting with constant a, students can derive the earlier equations of uniform motion. Stress constants of integration, which produce u in $v = u + at$ and the s_0 (s when $t = 0$) in $s = ut + \frac{1}{2}at^2 + s_0$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students can easily forget that if the velocity becomes negative, for example when a particle stops and changes direction, they need to split the integral to calculate distance rather than displacement.

NOTES

The following diagram may help students decide whether to differentiate or integrate to solve a problem. '**d**' for the **d**own arrow means '**d**ifferentiate'. Hence, down from 's' gives 'v' or $\frac{\mathrm{d}s}{\mathrm{d}t} = v$.

Integration is the opposite of differentiation so up is integrate, so up from 'a' gives 'v' or integral of 'a' with respect to t gives 'v'.

↓ s

diff ↓ v ↑ int

a ↑

Year 2: Remaining A Level Mathematics pure content
Pure Mathematics

Unit		Title	Estimated hours
1		**Proof:** Examples including proof by deduction* and proof by contradiction	3
2		**Algebraic and partial fractions**	
	a	Simplifying algebraic fractions	2
	b	Partial fractions	3
3		**Functions and modelling**	
	a	Modulus function	2
	b	Composite and inverse functions	3
	c	Transformations	3
	d	Modelling with functions* *examples may be Trigonometric, exponential, reciprocal etc.	2
4		**Series and sequences**	
	a	Arithmetic and geometric progressions (proofs of ‘sum formulae’)	4
	b	Sigma notation	2
	c	Recurrence and iterations	3
5		**The binomial theorem**	
	a	Expanding $(a + bx)^n$ for rational n; knowledge of range of validity	4
	b	Expansion of functions by first using partial fractions	3
6		**Trigonometry**	
	a	Radians (exact values), arcs and sectors	4
	b	Small angles	2
	c	Secant, cosecant and cotangent (definitions, identities and graphs); Inverse trigonometrical functions; Inverse trigonometrical functions	3
	d	Compound* and double (and half) angle formulae *geometric proofs expected	6
	e	$R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$	3
	f	Proving trigonometric identities	4
	g	Solving problems in context (e.g. mechanics)	2
7		**Parametric equations**	
	a	Definition and converting between parametric and Cartesian forms	3
	b	Curve sketching and modelling	2

Unit		Title	Estimated hours
8		**Differentiation**	
	a	Differentiating sin x and cos x from first principles	2
	b	Differentiating exponentials and logarithms	3
	c	Differentiating products, quotients, implicit and parametric functions.	6
	d	Second derivatives (rates of change of gradient, inflections)	2
	e	Rates of change problems* (including growth and kinematics) *see Integration (part 2) – Differential equations	3
9		**Numerical methods***	
	a	Location of roots	1
	b	Solving by iterative methods (knowledge of ‘staircase and cobweb’ diagrams)	3
	c	Newton-Raphson method	2
	d	Problem solving	2
		***See Integration (part 2) for the trapezium rule**	
10		**Integration (part 1)**	
	a	Integrating x^n (including when $n = –1$), exponentials and trigonometric functions	4
	b	Using the reverse of differentiation, and using trigonometric identities to manipulate integrals	5
11		**Integration (part 2)**	
	a	Integration by substitution	4
	b	Integration by parts	3
	c	Use of partial fractions	2
	d	Areas under graphs or between two curves, including understanding the area is the limit of a sum (using sigma notation)	4
	e	The trapezium rule	2
	f	Differential equations (including knowledge of the family of solution curves)	4
12		**Vectors (3D):** Use of vectors in three dimensions; knowledge of column vectors and **i**, **j** and **k** unit vectors	5
			120 hours

UNIT 1: Proof

Examples including proof by deduction and proof by contradiction (1.1)

Teaching time
3 hours

Return to overview

SPECIFICATION REFERENCES

1.1 Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction. Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs)

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

G20	Pythagoras Theorem
	Trigonometry
A18	Algebraic manipulation including completing the square
N4, N8, N10	Surds, prime and irrational numbers

AS Mathematics – Pure Mathematics content

1.1 Proof (See Unit 3a of the SoW)

KEYWORDS

Proof, verify, deduction, contradict, rational, irrational, square, root, prime, infinity, square number, quadratic, expansion, trigonometry, Pythagoras.

NOTES

Proof may also be tested throughout the specification through other topics e.g. trigonometry, series, differentiation, etc.

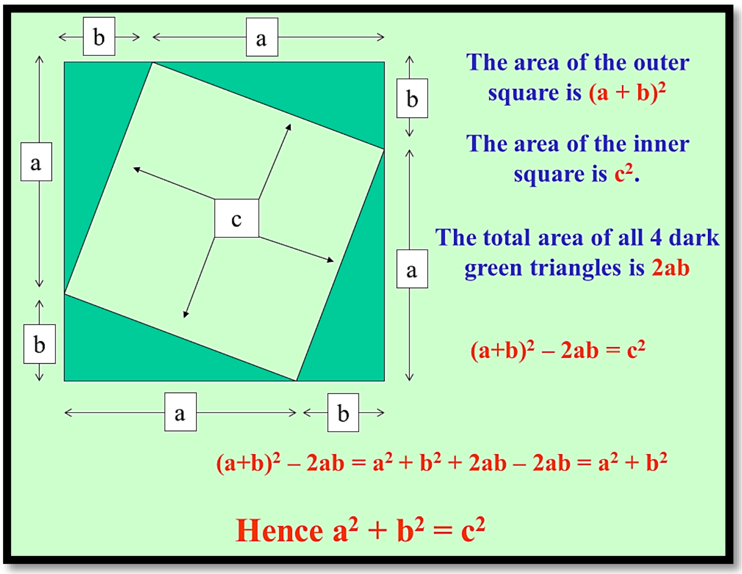
OBJECTIVES

By the end of the sub-unit, students should:

- understand that various types of proof can be used to give confirmation that previously learnt formulae are true, and have a sound mathematical basis;
- understand that there are different types of proof and disproof (e.g. deduction and contradiction), and know when it is appropriate to use which particular method;
- be able to use an appropriate proof within other areas of the specification later in the course.

TEACHING POINTS

Introduce using areas and the expansion of $(a + b)^2$ to prove Pythagoras' theorem as an example of using a logical sequence of steps in order to deduce a familiar result.



Explain how verification for a set number of values in *not* a proof of a general result (for all values of n).

Show how different methods can be used to prove a statement, including:

- Manipulating the LHS of a result and using logical steps (normally algebraic) to make it match the RHS or vice versa (or, sometimes, manipulating both sides to reach the same expression).
- Manipulating an expression to show it holds true for all values. For example, an inequality can always be ≥ 0 if we manipulate the LHS to be in the form of $[\text{something}]^2$ since anything squared will always be bigger or equal to zero. This argument can be used on a gradient function to prove a function is increasing.

Provide standard examples of proof by contradiction, e.g., $\sqrt{2}$ is irrational:

Assuming it can be written as a rational number $\frac{a}{b}$ which has been written in its lowest terms. It follows that $\frac{a^2}{b^2} = 2$ and $a^2 = 2b^2$. Therefore, a^2 is even because it is equal to $2b^2$. It follows that a must be even (as squares of odd integers are never even). Because a is even, there exists an integer k that fulfills: $a = 2k$. Substituting $2k$ for a above gives $2b^2 = (2k)^2 = 4k^2$, so $b^2 = 2k^2$. Because $2k^2 = b^2$, it follows that b^2 is even and b is also even.

Hence a and b are both even, which contradicts that $\frac{a}{b}$ is in its simplest form

Another example of proof by contradiction is the proof that there is an infinite number of primes:

Assume there is an integer p, such that p is the largest prime number.

Now $p! + 1 > p$ and is not divisible by p or any other number less than p*

*If 2 is a factor of n, then 2 is not a factor of $n + 1$. Similarly if 3 is a factor of n, 3 is not a factor of $n + 1$. Now 2, 3, … p are all factors of $p!$, so none are factors of $p! + 1$.

So, either $p! + 1$ is not divisible by an integer other than 1 or p! + 1 which means p! + 1 is prime, or $p! + 1$ is divisible by some number between p and $p! +1$ which implies there is a prime number larger than p.

These both contradict our initial assumptipn, which proves there are an infitie number of primes.

Illustrate proof by exhaustion e.g. Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$ for the positive integers from 1 to 5 inclusive.

This can be proved if you substitute (exhaust) all the possible values of n from 1 to 5. Note that this type of proof can only be used for proving something for a set of given values.

You should also talk about disproof by counter-example.

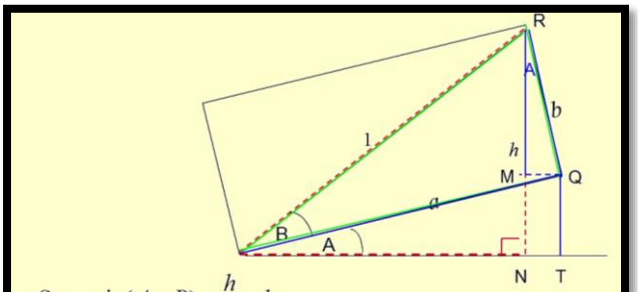
Explain that all we have to do is find *one* example where the statement does not hold and this is enough to show that it is not always true. This method can be used to disprove trigonometric identities as well as statements such as $a > b \Rightarrow a^2 > b^2$:

Choose any pair of negative numbers with $a > b$ e.g. $a = –2$ and $b = –3$.

Hence $a > b$, but if we square the numbers $a^2 < b^2$ (as $4 < 9$) and so this disproves the statement.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Link with Trigonometry (Unit 6d) and provide a deduction of the compound-angle formula for e.g. sin $(A + B)$



So, $\sin(A+B) = \dfrac{h}{1} = h$

But, h = NM + MR = TQ + MR

Now $TQ = a\sin A$ and $MR = b\cos A$

So, $h = a\sin A + b\cos A$

Also, $\sin B = b$ and $\cos B = a$

$\Rightarrow \sin(A+B) = \sin A\cos B + \cos A\sin B$

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some students mistakenly think that substituting several values into an expression is sufficient to prove the statement for all values.

Similarly, for example, referring to a graph to prove that the gradient is always positive rather than completing the square will not gain marks for a proof.

NOTES

Proof may be tested throughout the specification in other topics such as trigonometry, series, differentiation, etc.

UNIT 2: Algebraic and partial fractions

Return to overview

SPECIFICATION REFERENCES

2.6 Simplify rational expressions including by factorising and cancelling, and algebraic division (by linear expressions only)

2.10 Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than 3 terms, numerators constant or linear)

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A4 Algebraic fractions

AS Mathematics – Pure Mathematics content

2.6 Algebraic division, factor theorem (See Unit 3a of the SoW)

KEYWORDS

Polynomial, numerator, denominator, factor, difference of two squares, quadratic, power, index, coefficient, degree, squared, coefficients, improper, identity, algebraic fraction, partial fraction, rational.

NOTES

For algebraic fractions, denominators of rational expressions will be linear or quadratic,

e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$.

Partial fractions to include denominators such as:

$(ax + b)(cx + d)(ex + f)$ and $(ax + b)(cx + d)^2$.

This work has applications in A level Pure Mathematics topics such as series expansions (Unit 5), differentiation (Unit 8) and integration (Unit11).

2a. Simplifying algebraic fractions (2.6)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to add, subtract, multiply and divide algebraic fractions;
- know how to use the factor theorem to shown a linear expression of the form $(a + bx)$ is a factor of a polynomial;
- know how to use the factor theorem for divisors of the form $(a + bx)$;
- be able to simplify algebraic fractions by fully factorising polynomials up to cubic.

TEACHING POINTS

Revise the basic rules of numerical fractions and start with simplifying some GCSE (9-1) Mathematics algebraic fractions.

Exam questions tend to focus on factorising polynomials and then cancelling common factors to simplify algebraic fractions. For example:

Simplify $\frac{x^2-5x-6}{x^2-10x+24} \div \frac{x^2-x-2}{x^2-4x}$

You can use function notation when referring to fractions. (This has been covered in GCSE (9–1) Mathematics and also links with Unit 3.) For example:

The function f is defined by

$$\text{f}: x \longrightarrow \frac{3(x+1)}{2x^2+7x-4} - \frac{1}{x+4}, \quad x \in \mathbb{R},\ x > \frac{1}{2}$$

Show that $\text{f}(x) = \frac{1}{2x-1}$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

End this section by showing the reverse process where a simplified rational function is split into two (or more) partial factions. This links to the next set of lessons.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students need to practise factorising quadratics as this is often done incorrectly.

The most common errors include failing to include all necessary brackets, casual miswriting of signs part way through calculations and not dealing correctly with factors. Particular care with signs needs to be taken when a fraction follows a minus sign.

NOTES

Students must be able to divide polynomials for use in the partial fractions next.

2b. Partial fractions (2.10)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to split a proper fraction into partial fractions;
- be able to split an improper fraction into partial fractions, dividing the numerator by the denominator (by polynomial long division or by inspection).

TEACHING POINTS

Stress the fact that when we break-up a fraction into two or more partial fractions, we use an identity (≡) sign, and not an equal sign, as the expressions are equivalent for all values of x.

Start with a pair of algebraic fractions that need to be added together. Stress that the single fraction answer may be simplified, but that it can often be difficult to work with. For example in order to integrate the fraction it may be necessary to split it back up into two (or more) partial fractions. In other words, the reverse process from the previous section above needs to be carried out.

The number of partial fractions and the format of the individual terms, is dependent on two factors.

1. The maximum power (or degree) of the polynomials of the numerator and denominator. The degree of the denominator must be *greater* than that of the numerator. If the degree is equal or the degree of the numerator is greater (i.e. the fraction is improper), then algebraic division must be carried out first, and then the partial fractions formed.
2. The type and power of denominator.

   If the denominator is, e.g. $(x + 2)^2$, then we call this a *repeated* factor. In order to cover all possibilities of factors this has to be set up as two partial fractions with denominators $(x + 2)$ and $(x + 2)^2$.

   Show a numerical example with a denominator of 25, and hence the denominators of the partial fractions are 5 and 25.)

Examples of each of the following types need to be covered.

Linear: $\frac{5x-5}{(x+3)(x-2)}$ $\quad \frac{2}{x^2-1}$ $\quad \frac{7x+3}{x(x+1)}$

Repeated: $\frac{4x^2-3x+5}{(x-1)^2(x+2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Improper: $\frac{2x^2+5x-6}{(2x-1)(1+x)} \equiv A + \frac{B}{2x-1} + \frac{C}{1+x}$

As students work through examples, encourage them to experiment with the choice of values they substitute. If necessary remind them that $x = 0$ is an option. Also show that equating coefficients can sometimes be a more efficient alternative, sometimes avoiding the necessity for simultaneous equations.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Are there any values which make the denominators zero? Make links with the graphs of the functions and talk about how these values will correspond to exceptions and special cases in future topics where partial factions need to be found as a simplifying step.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some students will set up and solve simultaneous equations rather than using values of x to work out missing constants.

Ensure students are aware of the most efficient methods for solving different types of problem so they do not waste time in exam situations.

NOTES

The specification notes state, **‘**Denominators not more complicated than squared linear terms and with no more than 3 terms, numerators will be constant or linear’.

This unit has applications in Unit 5 – Series expansions, Unit 8 – Differentiation and Unit 11 – Integration.

UNIT 3: Functions and modelling

Return to overview

SPECIFICATION REFERENCES

2.7 The modulus of a linear function

2.8 Understand and use composite functions; inverse functions and their graphs

2.9 Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs:

$y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$

and combinations of these transformations

2.11 Use of functions in modelling, including consideration of limitations and refinements of the models

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A7 Vocabulary and f(x) notation for functions

A13 Composite, inverse and transformations of polynomial functions

A12 Knowledge of polynomial, trigonometric, exponential and logarithmic functions, including their graphs

AS Mathematics – Pure Mathematics content

2.9 Transforming graphs (See Unit 1f of the SoW)

KEYWORDS

Function, mapping, domain, range, modulus, transformation, composite, inverse, one to one, many to one, mappings, $\mathrm{f}(x)$, $\mathrm{fg}(x)$, $\mathrm{f}^{-1}x$, reflect, translate, stretch.

NOTES

This topic very much builds on the functions section of the GCSE (9-1) Mathematics specification. The exponential function is an important function for modelling real-world problems such as growth and decay etc.

3a. Modulus function (2.7)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand what is meant by a modulus of a linear function;
- be able to sketch graphs of functions involving modulus functions;
- be able to solve equations and inequalities involving modulus functions.

TEACHING POINTS

Define the modulus of a set of numbers as being the positive values only. e.g. $|-2| = 2$ and $|5| = 5$.

Begin by using an ICT graph-drawing package (either using the whiteboard or students' individual devices) to sketch some linear graphs using both $y =$ and $f(x) =$ notation, e.g. $y = 2x - 1$ or $f(x) = 2x - 1$.

Display the graph of $y = |2x - 1|$ and discuss this with students, drawing comparisons with the 'non-modulus' graph and making sure everyone recognises that $y = |2x - 1|$ does not have any negative values of y (the graph 'bounces up' with the x-axis acting like a mirror).

Define the term modulus function and use the general notation $y = |f(x)|$.

Ask students to predict what the graph of $y = 2|x| - 1$ will look like and then plot it. This time the values of x that are substituted into the function cannot be negative. In other words the graph on the left of the y-axis is a reflection of the graph on the right (where the x-values are positive) with the y-axis being the line of symmetry.

The general notation for this type of function is $y = f|x|$.

Students should be able to sketch the graphs of $y =|ax + b|$ and use their graphs to solve modulus equations and inequalities.

Use the graph-drawing package to sketch the graph of $y = |2x - 1|$ and $y = x$ and use these to solve $|2x - 1| = x$ by considering the points of intersection. Ask students to think about how they might solve this equation algebraically without using a graph. Solving $2x - 1 = x$ gives one solution, but how would the 'modulus' part be represented algebraically? What is the equation of the straight-line graph that represents the 'bounced' part which is now above the y-axis?

Extend this idea to looking at inequalities, for example how to solve $|2x - 1| > x$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

What happens if we square a modulus? $|-2|^2 = 4$, so a modulus squared is always positive.

Apply this to the modulus equation above $|2x - 1|^2 = x^2$; leading to $3x^2 - 4x + 1 = 0$.

This quadratic gives the two solutions to the equation $y = |2x - 1|$ above. Does this always work? Does this work for inequalities?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may find it difficult to sketch graphs involving modulus functions particularly if they are combined with other functions, for example logarithms.

In exam situations, often only the highest scoring students are able to solve modulus equations with x on both sides, or inequalities which involve the modulus function.

NOTES

The modulus function will also be used when expressing the validity of a binomial expansion in Unit 5.

3b. Composite and carte (2.8)	**Teaching time** 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to work out the domain and range of functions;
- know the definition of a one-one and a many-one mappings;
- be able to work out the composition of two functions;
- be able to work out the inverse of a function and sketch its graph;
- understand the condition for an inverse function to exist.

TEACHING POINTS

The notation f: $x \mapsto \ldots$ and f(x) will be used as in GCSE (9-1) Mathematics.

Students will need to understand exactly what functions are and the notation associated with them.

Domain and range from $\mathbb{R}$ (or a subset of $\mathbb{R}$) to $\mathbb{R}$ are important terms for students to understand and should be used regularly. Link this to function machines and graphs (where the domain is the set of x-values and the range is the set of corresponding y-values).

Students should be aware of one-one and many-one mappings and know that a function cannot be one-many.

Definitions and examples of odd and even functions will need to be given

Students need to know how to find the inverse of a function and it is worth stressing the notation here as lots of students still differentiate when they see this in an exam.

Students should know that if f^{-1} exists, then $ff^{-1}(x) = f^{-1}f(x) = x$. It follows from this that the inverse of a many-one function can only exist if its domain is restricted to make it a one-one function.

Composite functions are also introduced here and it is worth spending some time going over why the order is very important. Students must know that fg means 'do g first and then f'. It may be helpful to use an an additional set of brackets in the notation for composite functions, e.g. f[g(x)].

Draw lots of examples of the above using graphing packages and relate the mappings to the graphs. Give an example of a quadratic in which the range is determined by the minimum or maximum point.

Students must also know that the graph of $f^{-1}(x)$ is the image of the graph of $y = f(x)$ after reflection in the line $y = x$. You could relate this to the reverse function machine and the algebraic approach for finding an inverse function (when you change the subject of the formula and rewrite it in terms of x as the final step).

Ask questions such as:

When does the function machine fail to find an inverse?

Do any functions have a self-inverse?

Is an inverse function always possible?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The following activities are good for building familiarity and fluency with functions and notation.

1. Play a game where the teacher gives clues about a function and the student have to work out what the function is.

   Begin with numerical examples, e.g. $f(3) = 10$, $f(5) = 26$, $f(-2) = 5$ … until the correct $f(x)$ is given.

   Then to make it more relevant the clues should become algebraic, e.g. $f(4) = 14$, $f(2a) = 4a^2 - 2$, $f(x^2) = x^4 - 2$ etc.

2. Put all these ideas together by giving students two functions to explore. For example $f(x) = \frac{1}{x}$ and $g(x) = x + 1$ or $f(x) = |x|$ and $g(x) = x - 2$ or $f(x) = e^x$ and $g(x) = 2x - 1$.

   Students should explore the following using graphs etc.

   a) compare and contrast the graphs of $fg(x)$ and $gf(x)$
   b) work out if there are any one-one functions here
   c) find the inverses of any one-one functions (relating the inverses to the originals by sketching)

3. Students should investigate whether the following properties of functions are sometimes true, never true or always true.

   $fg(x) = gf(x)$ $\quad$ $g(x) = g^{-1}(x)$ $\quad$ $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$ $\quad$ $(fg)^{-1}(x) = f^{-1}g^{-1}(x)$

An extension activity could be to find as many functions as possible such that $fg(x) = gf(x)$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students can often successfully find the range in exam questions, but some give their answer in terms of x rather than $f(x)$.

When finding inverse functions, students need to remember to swap x and y. When describing why a function does not have an inverse, students should be advised to answer this question as “because it is not one to one” or “because it is many to one”.

NOTES

Relate and link this work on functions to exponentials and natural logarithms covered in AS Pure Mathematics (Unit 8).

3c. Transformations (2.9)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs and *combinations* of the transformations: $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$;
- be able to transform graphs to produce other graphs;
- understand the effect of composite transformations on equations of curves and be able to describe them geometrically.

TEACHING POINTS

Students should have some understanding of graph transformations from GCSE (9-1) Mathematics and AS Mathematics – Pure Mathematics, but this will not necessarily include combinations of transformations.

Students need to be able to sketch the transformations $y = a\mathrm{f}(x) + b$, $a\mathrm{f}(x + b)$ and $\mathrm{f}(ax) + b$, but will not be required to sketch $\mathrm{f}(ax + b)$

Use graph drawing packages to investigate the properties of familiar functions (such as trigonometric and exponential functions) when you apply the above transformations. Relate the geometry of the transformation to the algebra. For example, $\mathrm{f}(x) + a$ adds a to all the y-coordinates, hence the graph moves ‘up’ by a units (translation vector).

Pose the question, “Does the order in which transformations are applied matter?” Ask students to explore this and present their findings to the class.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can explore the difference between transforming x *before* it goes through the function and transforming it afterwards.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often score well on questions which involve describing geometrical transformations, but incorrect use of terminology will lose marks. Students must use the correct terms: stretch, scale factor and translation.

Students also need to be aware that the order of transformations is often important.

NOTES

Link with the work on transformations in AS Mathematics – Pure Mathematics, see SoW Unit 1f.

3d. Modelling with functions (2.11)

Teaching Time
3 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- use functions in modelling, including consideration of limitations and refinements of the models.

TEACHING POINTS

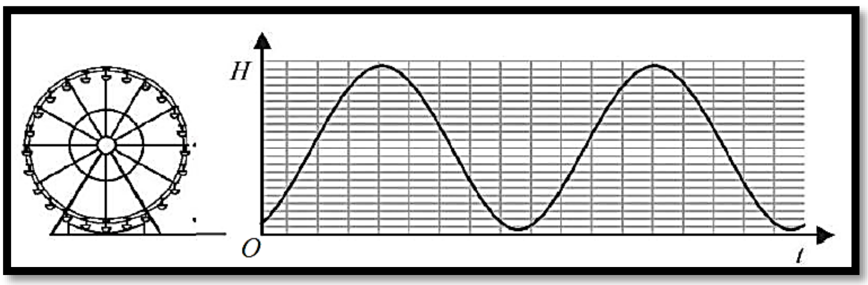
The specification gives some possible contexts in which functions can be used to model real-life situations. These are:

- Use of trigonometric functions for modelling tides, hours of sunlight, etc. (See the example in Unit 6g)
- Use of exponential functions for growth and decay (See AS Mathematics content - Pure Mathematics, Section 6.7).
- Use of reciprocal function for inverse proportion (e.g. Pressure and volume)

OPPORTUNITIES FOR REASONING/PROBLEM

Trigonometry example:

The height above the ground of a passenger on a Ferris wheel is modelled by the equation
$H = 11 - 10\cos(80t)° + 3\sin(80t)°$
where the height of the passenger above the ground is H metres, t minutes after the wheel starts turning.
Figure 3 below shows the graph of H against t for two complete cycles of the wheel.



Use the model to find the maximum height above the ground reached by the passenger

Exponential example:

The mass, m grams, of a radioactive substance t years after first being observed, is modelled by the equation

$$m = 25e^{-0.05t}$$

According to the model,

(a) state the value of m when the radioactive substance was first observed,

(b) show that $\frac{dm}{dt} = km$, where k is a constant that should be found.

(c) With reference to the model, interpret the significance of the sign of the value of k found in part (b).

NOTES

Rather like the 'Proof' (Unit 1), the applications of functions in context can appear throughout the specification. Link with 'Exponential Functions' (AS Mathematics - Pure Mathematics, see SoW Unit 8) & 'Trigonometry' in Unit 6.

UNIT 4: Series and sequences

Return to overview

SPECIFICATION REFERENCES

4.2 Work with sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$; increasing sequences; decreasing sequences; periodic sequences

4.3 Understand and use sigma notation for sums of series

4.4 Understand and work with arithmetic sequences and series, including the formulae for *n*th term and the sum to *n* terms

4.5 Understand and work with geometric sequences and series including the formulae for the *n*th term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r| < 1$; modulus notation

4.6 Use sequences and series in modelling

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A23 Generate terms of a sequence from either a term-to-term or a position-to-term rule

A24 Use simple arithmetic and geometric progression and geometric sequence

A25 Finding expressions for the *n*th term of linear and quadratic sequences

KEYWORDS

Sequence, series, finite, infinite, summation notation, Σ(sigma), periodicity, convergent, divergent, natural numbers, arithmetic series, arithmetic progression (AP), common difference, geometric series, geometric progression (GP), common ratio, *n*th term, sum to *n* terms, sum to infinity (S_∞), limit.

NOTES

Specification states: 'The proof of the sum formula should be known' and 'Given the sum of a series students should be able to use logs to find the value of *n*'. So this unit links to Unit 1 above (Proof) and to AS Mathematics – Pure Mathematics, see SoW Unit 8 (Exponentials and logarithms).

4a. Arithmetic and geometric progressions (proofs of 'sum formulae') (4.4) (4.5) (4.6)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know what a sequence of numbers is and the meaning of finite and infinite sequences;
- know what a series is;
- know the difference between convergent and divergent sequences;
- know what is meant by arithmetic series and sequences;
- be able to use the standard formulae associated with arithmetic series and sequences;
- know what is meant by geometric series and sequences;
- be able to use the standard formulae associated with geometric series and sequences;
- know the condition for a geometric series to be convergent and be able to find its sum to infinity;
- be able to solve problems involving arithmetic and geometric series and sequences;
- know the proofs and derivations of the sum formulae (for both AP and GP).

TEACHING POINTS

Start by recapping the work students did on sequences at GCSE (9-1) Mathematics before moving on to the new A level content, paving the way for the sigma notation in the following section.

Use practical situations, for example involving money, to illustrate APs and GPs and contrast the different ways they grow.

Find the *n*th term of a given arithmetic sequences and also use the rule to find the next two terms.

The Gauss problem (1 + 2 + … + 1000) is a good numerical way to lead into the full proof of the sum of an AP. Students will need to know the proof and derivation of the formula for the sum of an arithmetic sequence.

Illustrate how arithmetic sequences are different to geometric sequences, and explain that the common difference (a) becomes the common ratio (r). Students need to be aware that not all geometric sequences converge.

Cover problems where the n in the nth term formula (ar^{n-1}) is to be found using logarithms. (Show that it works if we use either base 10 or e.)

Illustrate when to use $\frac{a(1-r^n)}{(1-r)}$ and when to use $\frac{a(r^n-1)}{(r-1)}$ (depending on the value of r).

Show that $\frac{a}{(1-r)}$ can be derived if we illustrate on a calculator that r^n tends to zero when $-1 < r < 1$.

A way of illustrating the sum to infinity is to imagine hammering in a nail into a piece of wood, where each strike makes the nail sink in exactly half its remaining distance. There will be a limit to how many times it will need to be hit, as it surely will end up being 'flush' to the surface of the wood and have a distance of zero above the wood. (You can link this to Zeno's paradox.)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

This topic can be linked to mechanics by investigating, for example, a ball which is dropped from 2 m and bounces to $\frac{3}{4}$ of its height after each bounce.

Challenge students to come up with a rule to determine which series will have a sum to infinity and which won't.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When working with formulae for sequences and series, it is important that students state the relevant formula before substituting so that method marks can be awarded even if there is a numerical slip.

NOTES

Move onto general notation of series by using the sigma and recurrence notations in the next sessions.

4b. Sigma notation (4.3)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be familiar with $\sum$ notation and how it can be used to generate a sequence and series;
- know how this notation will lead to an AP or GP and its sum;
- Know that $\sum_{1}^{n} 1 = n$.

TEACHING POINTS

The key to understanding the concept of $\sum$ is to look at the limit values and substitute them into the nth term formula to generate the terms of the sequence.

Emphasise to students that they must take care when finding the starting point and never assume it starts with $n = 1$.

Students may initially find the $\sum$ notation tricky, particularly if they are not asked to find the sum of first n terms, but instead asked to find, e.g. the 7th to the 20th.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Challenge students to try to work out whether a sequence is an AP, GP or neither from just looking at the structure of the sigma version of a series?

Ask students to write a series in sigma notation

Show that $\Sigma n = \frac{1}{2}n(n + 1)$ is the sum of n natural numbers and relate this to the sum formula derived in the previous section.

Think about what to do if the upper limit is infinity.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

A fairly common error is to mix up the formulae for sums and terms, for example finding S_n rather than U_n and vice-versa.

NOTES

Students will need to be clear on the meanings and the usage of the various notations covered in this unit.

4c. Recurrence and iterations (4.2) (4.6)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know that a sequence can be generated using a formula for the *n*th term or a recurrence relation of the form $x_{n+1} = \mathrm{f}(x_n)$;
- know the difference between increasing, decreasing and periodic sequences;
- understand how a recurrence relation of the form $U_n = \mathrm{f}(U_{n-1})$ can generate a sequence;
- be able to describe increasing, decreasing and periodic sequences.

TEACHING POINTS

Work with sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$ and link this with the work done on iterations in GCSE (9-1) Mathematics.

Explore $x_{n+1} = \mathrm{f}(x_n)$ type series using graphics calculators or spreadsheets. (You can draw links between this work and Unit 9 – Numerical methods.)

Move on to general recurrence relations of the form $U_n = \mathrm{f}(U_{n-1})$ and investigate which sequences are increasing, decreasing and periodic. Spend some time looking at the different forms of notation for recurrence relations, making sure you cover examples of increasing, decreasing and periodic sequences. For example:

$u_n = \frac{1}{3n+1}$ describes a decreasing sequence as $u_{n+1} < u_n$ for all integers n

$u_n = 2^n$ is an increasing sequence as $u_{n+1} > u_n$ for all integers n

$u_{n+1} = \frac{1}{u_n}$ for $n > 1$ and $u_1 = 3$ describes a periodic sequence of order 2.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Cover questions in which sequences can be used to model a variety of different situations. For example, finance, growth models, decay, periodic (tide height for example) etc.

Can you tell from the structure of a recurrence relation how it will behave, and the type of sequence it will generate?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When asked to find the limit of u_n some candidates use the sum to infinity of a geometric series.

NOTES

Encourage the use of the ANS button on a calculator to obtain the terms for a recurrence relation.

UNIT 5: The binomial theorem

Return to overview

SPECIFICATION REFERENCES

4.1 Understand and use the binomial expansion of $(a + bx)^n$ for rational n, including its use for approximation;

be aware that the expansion is valid for $\left|\frac{bx}{a}\right| < 1$ (proof not required)

PRIOR KNOWLEDGE

Covered so far

- Series and sequences (See Unit 4 of the SoW)

GCSE (9-1) in Mathematics at Higher Tier

A4 Algebraic fractions

AS Mathematics – Pure Mathematics content

2.6 Algebraic division, factor theorem (See Unit 3a of the SoW)

4.1 Binomial expansion of the form $(a + bx)^n$, where n is a positive integer (See Unit 3b of the SoW)

KEYWORDS

Binomial, expansion, theorem, integer, rational, power, index, coefficient, validity, modulus, factorial, ${}^nC_r$, combinations, Pascal's triangle, partial fractions, approximation, converges, diverges, root.

NOTES

The formula book includes formulae for the binomial expansion:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times \ldots \times r}x^r + \cdots \qquad (|x| < 1, n \in \mathbb{R})$$

This unit links with the binomial distribution in the statistics section of the remaining A level Mathematics content of statistics.

5a. Expanding $(a + bx)^n$ for rational n; knowledge of range of validity (4.1)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the binomial expansion of $(1 - x)^{-1}$ for rational values of n and $|x| < 1$;
- be able to find the binomial expansion of $(1 + x)^n$ for rational values of n and $|x| < 1$;
- be able to find the binomial expansion of $(1 + bx)^n$ for rational values of n and $|x| < \frac{1}{|b|}$;
- be able to find the binomial expansion of $(a + x)^n$ for rational values of n and $|x| < a$;
- be able to find the binomial expansion of $(a + bx)^n$ for rational values of n and $\left|\frac{bx}{a}\right| < 1$;
- know how to use the binomial theorem to find approximations (including roots).

TEACHING POINTS

Begin by reviewing the expansion of $(a + b)^n$ when n is a positive integer.

Ask students to expand $(1 + x)^4$ and then try $(1 + x)^{-2}$. Why does it fail to work? Which coefficient calculation breaks down?

Explain how the binomial theorem allows us to expand *any* power. (Explain the reasoning behind the factorial notation using the explanation in the *Reasoning and problem solving* section below.)

Consider why the expansions are infinite when the power is *not* a positive integer. How far do we need to expand and to which term? (For example, up to and including coefficients of x^3.)

Take care to show the precision needed when dealing with negative calculations by demonstrating examples such as $(1 - 2x)^{-\frac{1}{2}}$.

If we expanded $(1 + x)^{\frac{1}{2}}$ then substituted $x= -0.1$, we would be effectively finding the square root of 0.9. Ask students to use a calculator to find an accurate value for $\sqrt{9}$. How many terms of the expansion would we need to substitute into in order to get a 4 decimal place version of the accurate value?

What happens when we substitute $x = 3$? Does this find the square root of 4?

Explain that if we raise a number > 1 to a positive power, it 'grows' and diverges out of control. This means that the value of x must be such that $-1 < x < 1$ or $|x| < 1$ in order to use the expansion of $(1 + x)^n$. The validity of the expansion is dependent upon the value of x we substitute into the terms.

Cover examples that build-up the expansions listed in the objectives above, ending with $(a + bx)^n$ for rational values of n and valid for $\left|\frac{bx}{a}\right| < 1$.

Introduce the concept of expansions of expressions which start with a rather than 1. Begin by showing that if we have $(2 + x)$ and if we want to make this start with a 1 in the bracket, we must take out the factor of 2, giving $2(1 + \frac{x}{2})$.

Now show for example, that $2(1 + 4)$ gives the same result as $(2 + 8)$ if we multiplied this out, but that if the bracket were squared the result would not be the same i.e. $2(1 + 4)^2 \neq (4 + 8)^2$.

However, $2^2(1 + 4)^2 = (4 + 8)^2$, so we need to raise the factor to the same power of the bracket and $(a + bx)^n = a^n(1 + \frac{bx}{a})^n$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Show how ${}^nC_r$ will only work on a calculator for positive integer values of n (as was done in Unit 3 of Pure).

However, we can instead use the definition and formula for selections (shown here for 'choose 2 from n different objects').

$$
\begin{aligned}
{}^nC_2 &= \frac{n!}{2!\,(n-2)!} \\
&= \frac{n(n-1)\cancel{(n-2)}\cancel{(n-3)}\ldots}{2!\,\cancel{(n-2)}\cancel{(n-3)}\ldots} \\
&= \frac{n(n-1)}{2!}
\end{aligned}
$$

This formula works for *all* values of n and follows the pattern of the binomial theorem as stated in the formula book.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When expanding $(1+4x)^{\frac{1}{2}}$ most students got the first two terms of the expansion correct, but often there was a mistake in the x^2 term, with $4x$ becoming just x being the common error. Some students made arithmetic errors with 4^2, by failing to actually square the 4, and others failed to simplify the binomial coefficient correctly.

When expanding an expression of the form $(a + x)^n$ a common error is to write this as $a(1 + \frac{x}{a})^n$ rather than $a^n(1 + \frac{x}{a})^n$.

Other errors include algebraic errors when combining two expansions, doing more work than is necessary when, for example, only terms up to x^2 are required, including the equality in the expression for the range of valid values for x and lack of understanding when using the modulus symbol (writing expressions such as $|x| < -4$).

NOTES

Link this section to the next part of this unit: expanding functions by first using partial fractions.

5b. Expansion of functions by first using partial fractions (4.1)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use partial fractions to write a rational function as a series expansion.

TEACHING POINTS

This sub-unit links with sub-unit 2b above (Partial Fractions) and gives the students a purpose for learning how to break-up a rational function into two or more partial fractions.

If we consider the 'complicated' fraction below, it needs to be simplifies into two simpler fractions each of which only involve a *single* algebraic bracket.

$$\frac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

We can now rewrite each term as a binomial series. (It is important to demonstrate that the $\frac{B}{x-1}$ term will become $B(x - 1)^{-1}$.)

Particular care needs to be taken when working with brackets that don't start with 1, and also when multiplying out all the terms to arrive at the final simplified series (up to and including the power required).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

You will need to assess all the separate validities for the individual binomial terms to declare the validity for the final series.

Include examples in which one of the terms is not a binomial and just multiplies without expansion. For example, $(2x - 1)(1 + 3x)^{-2}$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

(These all relate directly to the example given above.)

Nearly all students were able to make the connection between the parts of the question, but there were many errors in expanding both $(x - 1)^{-1}$ and $(2 + x)^{-1}$.

Few were able to write $(x - 1)^{-1}$ as $-(1 - x)^{-1}$ and the resulting expansions were incorrect in the majority of cases, both $1+x-x^2$ and $1-x-x^2$ being common errors.

However, $(2 + x)^{-1}$ was handled better, but the constant $\frac{1}{2}$ in $\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$ was frequently incorrect.

NOTES

Inform the students that partial fractions are also required to break down rational functions before they are differentiated (Unit 8) and integrated (Units 10 and 11).

UNIT 6: Trigonometry

Return to overview

SPECIFICATION REFERENCES

5.1 Work with radian measure, including use for arc length and area of sector

5.2 Understand and use the standard small angle approximations of sine, cosine and tangent i.e. $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$ where θ is in radians

5.3 Know and use exact values of sin and cos for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π and multiples thereof, and exact values of tan for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π and multiples thereof

5.4 Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding of their graphs; their ranges and domains

5.5 Understand and use $\sec^2\theta = 1 + \tan^2\theta$ and $\text{cosec}^2\theta = 1 + \cot^2\theta$

5.6a Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$; understand geometrical proofs of these formulae

5.6b Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

5.8 Construct proofs involving trigonometric functions and identities

5.9 Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

G22 Sine and cosine function

G18 Length of arc and area of sector

AS Mathematics – Pure Mathematics content

2.6 Algebraic division, factor theorem (See Unit 3a of the SoW)

5.7 Solving trigonometric equations (See Unit 4 of the SoW)

5.5 $\sin^2 x + \cos^2 x = 1$ and $\frac{\sin x}{\cos x} = \tan x$ (See Unit 4 of the SoW)

5.3 Properties of graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ (See Unit 4 of the SoW)

KEYWORDS

Pythagoras, Pythagorean triple, right-angled triangle, opposite, adjacent, hypotenuse, trigonometry, sine, cosine, tangent, secant, cosecant, cotangent, SOHCAHTOA, exact, symmetry, periodicity, identity, equation, interval, quadrant, degree, radian, circular measure, infinity, asymptote, small angles, approximation, identity, proof.

NOTES

This unit is fundamental to future study of trigonometry in Further Maths and also links to mechanics. For example, the path of a projectile requires the identity $1 + \tan^2 x = \sec^2 x$.

6a. Radians (exact values), arcs and sectors (5.1) (5.3)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the definition of a radian and be able to convert between radians and degrees;
- know and be able to use exact values of sin, cos and tan;
- be able to derive and use the formulae for arc length and area of sector.

TEACHING POINTS

Ensure all students know how to change between radian and degree mode on their own calculators and emphasise the need to check which mode it is in.

Radian measure will be new to students and it is important that they understand what 1 radian actually is.

Make sure students know that 'exact value' implies an answer must be given in surd form or as a multiple of π. They need to know the exact values of sin and cos for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π (and their multiples) and exact values of tan for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π (and their multiples).

Emphasise the need to always put a scale on both axes when drawing trigonometric graphs; students must be able to do this in radians.

Make links between writing the trig ratio of any angle (obtuse/reflex/negative) to the trig ratio of an acute angle and to the trig graphs. (Do not rely on the CAST method as this tends to show a lack of understanding.)

Derive the formulae for arc length and area of a sector by replacing the $\frac{\theta}{360°}$ in the GCSE formulae with $\frac{\theta}{2\pi}$.

The πs cancel giving length of arc = $r\theta$ and area of sector = $\frac{1}{2}r^2\theta$.

Cover examples which will involve finding the area of a segment by subtracting a triangle from a sector.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

One radian can be defined as 'the angle at the centre of a circle which measures out exactly one radius around the circumference.' Therefore, using $C = 2\pi r$, we can conclude that the full circumference, C is made up of 2π radians. This means 360 is equivalent to 2π radians.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

A common exam mistake is for students to have their calculators set in the wrong mode resulting in the loss of accuracy marks.

6b. Small angles (5.2)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the standard small angle approximations for sine, cosine and tangent.

TEACHING POINTS

The Specification states:- $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$ (where θ is in radians)

Experiment with trigonometric graphs and a graph-drawing package by reading off values near the origin and zooming into small angles so the students get a feeling for this new concept.

The formal proof is based on considering the area of a sector in which the angle is so small, the shape becomes a right-angled triangle (since the curved part is straightened).

By considering the area of the triangle within the sector, the area of the sector and the area of the right angled triangle we can see that

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

Cancelling $\frac{1}{2}r^2$ gives $\sin\theta < \theta < \tan\theta$

Dividing by $\sin\theta$ gives $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

As θ tends to 0, $\frac{1}{\cos\theta}$ tends to 1, and so $\frac{\theta}{\sin\theta}$ must tend to 1 as it is fixed between two values which tend to 1.

So $\frac{\theta}{\sin\theta}$ is approximately equal to 1 for small values of θ (the small angle was the assumption at the start).

Rearranging gives $\sin\theta \approx \theta$.

Following a similar process, but dividing by $\tan\theta$ at the start gives $\tan\theta \approx \theta$.

Using the identity $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$ (which is covered in a later sub-unit), and substituting $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$, gives the third approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$.

The small angle approximations can be used to give estimated values of trigonometric expressions. For example, $\frac{\cos 3x - 1}{x\sin 4x}$ approximates to $-\frac{9}{8}$ (when x is small)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

These approximations only work when the small angles are measured in radians. Why don't the approximations work in degrees?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may try to use these approximations when angles are measured in degrees rather than radians.

NOTES

Small angles are also used in further mechanics work when dealing with simple pendulums.

6c. Secant, cosecant and cotangent (definitions, identities and graphs) & inverse trigonometrical functions & inverse trigonometrical functions (5.4) (5.5)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the secant, cosecant and cotangent functions, and their relationships to sine, cosine and tangent;
- be able to sketch the graphs of secant, cosecant and cotangent;
- be able to simplify expressions and solve involving sec, cosec and cot;
- be able to solve identities involving sec, cosec and cot;
- know and be able to use the identities $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \text{cosec}^2 x$ to prove other identities and solve equations in degrees and/or radians
- be able to work with the inverse trig functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$;
- be able to sketch the graphs of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

TEACHING POINTS

Introduce students to the reciprocal trigonometric functions secant θ, cosecant θ and cotangent θ.

A good way to introduce these as reciprocal trig functions is to start by asking whether there is another way of writing x^{-1}. This should lead to the answer $\frac{1}{x}$. If we try this with $\sin^{-1}\theta$ it is not the same meaning as $\frac{1}{\sin\theta}$, so we need to name a different function cosec θ. (Contrast this with inverse trig functions looked at later in this section)

To help students remember which reciprocal function goes with sin, cos and tan, point out that the **third** letter of these new functions, gives the name of the trig function in the denominator, i.e.

se**c** $\theta = \frac{1}{\cos\theta}$ co**s**ec $\theta = \frac{1}{\sin\theta}$ co**t** $\theta = \frac{1}{\tan\theta}$

You should also point out that cot θ can be written as the reciprocal of tan θ to give $\frac{\cos\theta}{\sin\theta}$.

Students will be expected to know what the graphs of each of the reciprocal and inverse functions look like and their key features, including domains and ranges. The relationships between the graphs and their originals can be explored on graphical calculators or graphing Apps.

Show students how to work out new trigonometric identities by dividing $\sin^2\theta + \cos^2\theta = 1$ (from AS Mathematics – Pure Mathematics) by $\cos^2\theta$ or by $\sin^2\theta$ to give the two new identities: $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \text{cosec}^2\theta$.

This is a good alternative to simply remembering the identities and lessens the chance of mixing them up.

It is a good idea to use the new identities to solve trigonometric equations (which are often quadratic look-a-likes) before proving identities. Sub-unit 6f covers proving identities when all the available formulae have been covered.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

To contrast **reciprocal** trig functions students will also need to be familiar with the **inverse** functions of $\sin\theta$,$\cos\theta$ and $\tan\theta$. They will again need an understanding of the graphs of $\arcsin\theta$, $\arccos\theta$ and $\arctan\theta$. Refer back to the work on functions and emphasise that for arcsin, arccos and arctan to be true functions there must be a one-one relationship between domain and range and so the domains must be restricted to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors in these questions involve using wrong notation, for example $\sin x^2$ instead of $\sin^2 x$, or making algebraic mistakes. Students sometimes struggle to deal with more complicated functions such as $\text{cosec}\,(3x + 1)$ and do not always recognise where trigonometric identities can be used.

NOTES

These trigonometric functions will be useful tools for the calculus units that follow later in the course.

6d. Compound* and double (and half) angle formulae (5.6a) *geometric proofs expected	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to prove geometrically the following compound angle formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$;
- be able to use compound angle identities to rearrange expressions or prove other identities;
- be able to use compound angle identities to rearrange equations into a different form and then solve;
- be able to recall or work out double angle identities;
- be able to use double angle identities to rearrange expressions or prove other identities;
- be able to use double angle identities to rearrange equations into a different form and then solve.

TEACHING POINTS

A good introduction is to ask the class to work out $\sin(30 + 60)°$.It is equal to $\sin(90)° = 1$. Go on to ask whether $\sin 30° + \sin 60°$ gives the same value (either using a calculator or using surds). They should discover that the values are different. Explain that the reason for this is that you can't simply multiply out functions in this way.

This leads in to explaining why compound angle formulae are needed to calculate $\sin(A + B)$.

Unit 1 above gives an example of a geometric proof by deduction for $\sin(A + B)$.

Care needs to be taken when using the result to extend to $\sin(A – B)$ for negative values. Students will need to remember that $\cos(–B) = \cos B$ and that $\sin(–B) = –\sin(B)$.

Extend these formulae by substituting $A = B$ to derive the double angle formulae

Show that there is only one version of $\sin 2x = 2 \sin x \cos x$, but the basic version of $\cos 2x = \cos^2 x – \sin^2 x$, can be re-written by substituting $\cos^2 x + \sin^2 x = 1$ (from AS Mathematics – Pure Mathematics) into two different versions (exclusively in $\sin x$ or $\cos x$).

A critical part of future questions and proofs involves choosing the correct version of the compound and/or double angle formulae.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Derive and cover examples using half angle formulae by adapting the double angle versions.

The next sub- unit will look at how to solve equations of the type $a \cos\theta + b \sin\theta = C$, using compound angles to rewrite and simplify the expression on the left hand side.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors are sign errors when using the compound and double angle formulae.

NOTES

$t\,(\tan\frac{1}{2}\theta)$ formulae will *not* be required.

You should cover reading off obtuse and reflex values by considering a right-angled triangle and assigning a negative or positive sign depending on which quadrant the angle lies in.

Double angle formulae will be a vital substitution when presented in calculus later in the course

6e. R cos ($x \pm \alpha$) or R sin ($x \pm \alpha$) (5.6b)	**Teaching time** 3 hours

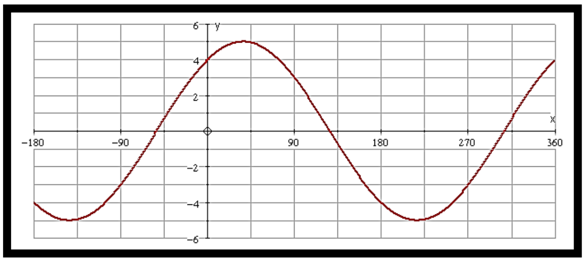
OBJECTIVES

By the end of the sub-unit, students should:

- be able to express $a\cos\theta + b\sin\theta$ as a single sine or cosine function;
- be able to solve equations of the form $a\cos\theta + b\sin\theta = c$ in a given interval.

TEACHING POINTS

Start by drawing a graph of, say, 4 cos x + 3 sin x to show that is has the basic sin–cos shape. Where are the coordinates of the maximum or minimum points? It approximately fits 5 cos (x – 40°).



Equating 4 cos x + 3 sin x to an expanded form of R cos (x – α) gives:

$$4\cos x + 3\sin x \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$$

Equating coefficients leads to:

$$R\sin\alpha = 3 \text{ and } R\cos\alpha = 4.$$

By squaring and adding we obtain $R = 5$, and by dividing we obtain $\alpha = 36.9°$. (This confirms the approximate fit above.)

Move on to solving equations of the type $a\cos\theta + b\sin\theta = c$ using R cos ($x \pm \alpha$) or R sin ($x \pm \alpha$) as the first step. Effectively, the question reduces to a trigonometry equation like those done in Pure Paper 1, but at this level the angles could be in radians.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Ask students whether they can relate the R and α to the basic properties of the curve.

Think about the maximum/minimum value and where it occurs.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiner comments suggest that the part of the calculation which causes most problems is working out the angle α:

When writing $a\cos\theta + b\sin\theta$ into the form $R\sin(\theta - \alpha)$ most students found the value of R correctly, the same was not true of the angle α. Some students seemingly failed to notice that α was given as an acute angle.

When solving an equation of the form $a\cos\theta + b\sin\theta = c$ many students seemingly could not cope with the result of –39.23° that their calculator gave them and could not get the first solution. In addition some students found the third quadrant solution only, whereas some found more than two solutions. However many students did give a fully correct solution, often by using a sketch graph to help them decide where the solutions lay.

NOTES

On the legacy specifications, the form of expression to use was given in the question. Encourage students to choose which form to use. It is better to choose the version which, when expanded, gives the same signs for the corresponding terms as the original expression.

6f. Proving trigonometric identities (5.8)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to construct proofs involving trigonometric functions and previously learnt identities.

TEACHING POINTS

Proving trigonometric identities is something that challenges many students and is considered by some to be the most challenging part of the course.

The basic principles are the same as in Unit 1 (Proof): manipulate the LHS and use logical steps to make it to match the RHS or vice-versa. (Sometimes both sides can be manipulated to reach the same expression.)

Make sure you explain why we use $\equiv$ rather than $=$.

In the example below, the most efficient method is to start with the LHS and use $\sec^2\theta = 1 + \tan^2\theta$ to replace the numerator. The vital step is to multiply top and bottom of the resulting fraction by $\cos^2\theta$, this leads to the two familiar identities involving $\sin^2\theta$ and $\cos^2\theta$.

(a) Prove that

$$\frac{\sec^2\theta}{1-\tan^2\theta} \equiv \sec 2\theta, \qquad \theta \neq \frac{n\pi}{4}, n \in \mathbb{Z}$$

(b) Hence state a reason why the equation

$$\frac{\sec^2\theta}{1-\tan^2\theta} = \frac{1}{2}$$

does not have any solutions.

The final step has 'Hence', so students should be encouraged to use the result in part (a) and write $\sec 2\theta = \frac{1}{2}$, which leads to $\cos 2\theta = 2$.

Students now need to explain fully that $-1 \leq \cos 2\theta \leq 1$, and so $\cos 2\theta = 2$ has no solutions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The specification says 'Students need to prove identities such as $\cos x \cos 2x + \sin x \sin 2x \equiv \cos x$.'

Sub-unit 6g gives some examples of where trigonometry is used for problem solving.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

These questions often prove to be the most demanding on the paper and serve to differentiate between students.

Students need to make sure they include all steps in the proof with full explanation.

NOTES

It is a essential that students know which formulae are provided in the formulae book and which have to be learnt.

6g. Solving problems in context (e.g. mechanics) (5.9)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.

TEACHING POINTS

Links can be made with simple harmonic motion in further mechanics, where a sin and/or cos curve could model the height of the tide against a harbour wall. When is it safe for the ship to come into the port?

For kinematics the velocity equation could be expressed as $v = 3\sin(2t)$ m s^{-1}. The times at which the object is stationary or at maximum speed could be analysed (no calculus at this stage).

An oscillating share price could be modelled using trigonometric equations. Ask students: when is the best time to buy and sell?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

$h = a\cos t + b\sin t$ will model the tide height, h, and makes a good link with the previous section. ($t \geq 0$) Is t in degrees or radians?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The following question and comments come from a paper set in June 2014.

> A student records the number of hours of daylight each Sunday throughout the year. She starts on the last Sunday in May with a recording of 18 hours, and continues until her final recording 52 weeks later. She models her results with the continuous function given by
>
> $$H = 12 + 6\cos\left(\frac{2\pi t}{52}\right) + 2.5\sin\left(\frac{2\pi t}{52}\right), \quad 0 \leq t \leq 52$$
>
> where, H is the number of hours of daylight and t is the number of weeks since her first recording.
>
> Use this function to find the maximum and minimum values of H predicted by the model.

This was probably the least successful question. Although a good number of students could write down the maximum and minimum easily, some of those who had a correct value for the maximum then gave either –18.5 or 12 as the minimum. There were quite a number of students who had no idea how to tackle this part, often using values of H when $t = 0$ and $t = 52$.

Many students showed very little working at this stage, so it was sometimes unclear how much of the work was accurate.

NOTES

The specification says: ‘Problems could involve (for example) wave motion, the height of a point on a vertical circular wheel, or the hours of sunlight throughout the year. Angles may be measured in degrees or in radians.’

UNIT 7: Parametric equations

Return to overview

SPECIFICATION REFERENCES

3.3 Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms

3.4 Use parametric equations in modelling in a variety of contexts

PRIOR KNOWLEDGE

Covered so far

- Trigonometric identities
- Knowledge of a variety of functions involving powers, roots, trigonometric functions, exponentials and logarithms

GCSE (9-1) in Mathematics at Higher Tier

G11 Coordinate geometry

A2, A5 Changing the subject of the formula, and substitution

A12 Graphs of linear, quadratic and trigonometric functions

AS Mathematics – Pure Mathematics content

2.7, 3.1, 3.2 Coordinate geometry (See Unit 2 of SoW)

5.5, 5.7 Trigonometric identities (See Unit 4b of SoW)

KEYWORDS

Parametric, Cartesian, convert, parameter *t*, identity, eliminate, substitute, circle, hyperbola, parabola, ellipse, domain, modelling.

NOTES

Later in the course, students will need to be able to differentiate (using the chain rule) parametric equations to find tangents, normals, turning points etc.

7a. Definition and converting between parametric and Cartesian forms (3.3)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the difference between the Cartesian and parametric system of expressing coordinates;
- be able to convert between parametric and Cartesian forms.

TEACHING POINTS

Begin by explaining the difference between the Cartesian system, when a graph is described using $y = \mathrm{f}(x)$, and the parametric system, which uses $x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$ for some parameter t.

Illustrate this by asking the class to consider $x = 5t$ and $y = 3t^2$ and to try to eliminate t from the two equations. This will give $y = \frac{3}{25}x^2$ or $25y = 3x^2$. (This is a quadratic equation – parabola.)

Repeat for $x = 5t$ and $y = \frac{5}{t}$. This becomes $y = \frac{25}{x}$ (a hyperbola).

Sometimes we need to eliminate the parameter, t, by using identities rather than substitution.

Consider $x = 3\cos t$ and $y = 3\sin t$. Squaring both equations and adding means we can use $\cos^2 t + \sin^2 t = 1$ to give $x^2 + y^2 = 9$. (This is a circle, centre (0, 0) of radius 3.)

Ask students to use similar methods to show that $x = 2 + 5\cos t,\ \ y = -4 + 5\sin t$ describes a circle centre (2, −4) with radius 5.

How do we convert from Cartesian to parametric? (We need to be in radians) For example, what are the pair of parametric equations for a circle, centre (3, 5) radius 10?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

What shape is given by $x = 4\cos t$, $y = 2\sin t$?

Name and properties of curve? (See sub-unit 7b for plotting.)

The trigonometric identities in Unit 6 (such as $\sec^2 x = 1 + \tan^2 x$) can be used to convert from parametric to Cartesian form.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may have difficulties making any progress with these sorts of questions if they cannot work out which trigonometric identity to apply when eliminating the parameter t.

NOTES

The next section will look at how to plot parametric equations and modelling examples.

7b. Curve sketching and modelling (3.3) (3.4)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to plot and sketch curves given in parametric form;
- recognise some standard curves in parametric form and how they can be used for modelling.

TEACHING POINTS

It is often easier to match the properties of a curve in parametric form than it is in its Cartesian form.

In order to establish the shapes of some well-known curves such as circles, ellipses etc., ask the students to plot the pair of parametric equations in the form of a table of values.

When plotting $x = 4 \cos t$, $y = 4 \sin t$ what will the range of t be? (Remember to use radians.)

Now plot $x = 4 \cos t$, $y = 2 \sin t$. (This is the shape mentioned in the reasoning/problem solving section of sub-unit 7a.)

What values of t will we need for $x = 5t$, $y = \frac{5}{t}$?

Investigate parametric equations which give closed loops. These will be integrated later in course to find the area of a loop, so we need to establish how values of t link plotting (direction vital).

The specification states 'Students should pay particular attention to the domain of the parameter t, as a specific section of a curve may be described.'

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

A shape may be modelled using parametric equations (e.g. an object moves with constant velocity from (1, 8) at $t = 0$ to (6, 20) at $t = 5$), or students may be asked to find parametric equations for a motion.

Make links to Unit 10 (Kinematics) of Applied Paper 3.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The examiner comments for these questions illustrate how difficult students find this topic:

The final part proved very demanding and only a minority of students were able to use one of the trigonometric forms of Pythagoras to eliminate t and manipulate the resulting equation to obtain an answer in the required form.

Few even attempted the domain and the fully correct answer $0° \leq t \leq 2\pi$, was very rarely seen.

NOTES

Parametric equations is assumed knowledge for the calculus work in Further Mathematics – Further Pure Mathematics where students must find the volume of revolution for a solid formed by a pair of parametric equations

UNIT 8: Differentiation

Return to overview

SPECIFICATION REFERENCES

7.1c Differentiation from first principles for $\sin x$ and $\cos x$

7.1b Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection

7.2 Differentiate e^{kx}, a^{kx}, $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples. Understand and use the derivative of $\ln x$

7.4 Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions

7.5 Differentiate simple functions and relations defined implicitly or parametrically, for first derivative only

7.6 Construct simple differential equations in pure mathematics and in context, (contexts may include kinematics, population growth and modelling the relationship between price and demand)

PRIOR KNOWLEDGE

Covered so far

- Functional notation including $f'(x)$

GCSE (9-1) in Mathematics at Higher Tier

G11 Coordinate geometry

A2, A6 Changing the subject of the formula, and substitution

A12 Graphs of linear, quadratic and trigonometric functions

AS Mathematics – Pure Mathematics content

2.7, 3.1, 3.2 Coordinate geometry (See Unit 2 of SoW)

5.5, 5.7 Trigonometric identities (See Unit 4b of SoW)

7 Differentiation (See Unit 6 of SoW)

KEYWORDS

Derivative, tangent, normal, turning point, stationary point, maximum, minimum, inflexion, parametric, implicit, differential equation, rate of change, product, quotient, first derivative, second derivative, increasing function, decreasing function.

NOTES

This topic builds on the differentiation covered in AS Mathematics – Pure Mathematics, see SoW Unit 6 and leads into integration.

8a. Differentiating sin *x* and cos *x* from first principles (7.1c)

Teaching time
2 hours

OBJECTIVES

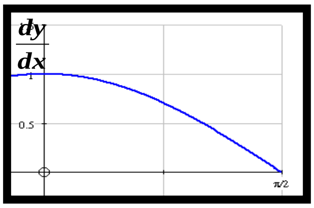
By the end of the sub-unit, students should:

- be able to find the derivative of $\sin x$ and $\cos x$ from first principles.

TEACHING POINTS

Review how to differentiate polynomials from first principles.

Sketch $y = \sin x$ and consider the gradient at key points by looking at slopes of tangents. If we plot the gradients then we get a shape which looks like the start of a cos graph:



This suggests that if $y = \sin x$, then $\frac{dy}{dx} = \cos x$, but this is not a proof or derivation!

Approach the differentiation from first principles in the same way as in AS Mathematics – Pure Mathematics, see SoW Unit 6.

Let's take a chord for $y = \sin x$ at $(x, \sin x)$ and $(x + \delta x, \sin(x + \delta x))$, the gradient of the chord is $\frac{\sin(x+\delta x)-\sin x}{\delta x}$

Using compound angle identity for sin $(A + B)$ we find that $\frac{\sin x\cos\delta x+\cos x\sin\delta x-\sin x}{\delta x}$

By manipulation we obtain $\frac{\sin x(\cos\delta x-1)}{\delta x} + \cos x\frac{\sin\delta x}{\delta x}$

Since $\delta x \to 0$, $\frac{\sin\delta x}{\delta x} \to 1$ and $\frac{\cos\delta x-1}{\delta x} \to 0$ we conclude that $\lim_{\delta x\to 0}\frac{\sin(x+\delta x)-\sin x}{\delta x} = \cos x$

Therefore the gradient of the chord → gradient of the curve and we conclude that $\frac{dy}{dx} = \cos x$.

A similar argument with $y = \cos x$ as a starting point leads to:

$$\frac{\cos(x+\delta x)-\cos x}{\delta x} = \frac{\cos x\cos\delta x-\sin x\sin\delta x-\cos x}{\delta x}$$

and therefore finding the derivative to be $-\sin x$.

The alternative notations $h \to 0$ rather than $\delta x \to 0$ are acceptable.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Ask the students to experiment with a graph-drawing package to verify that the gradient functions of $\sin x$ and $\cos x$ match the result found using first principles. Students must understand that the differentiation of $\sin x$ and $\cos x$ can only be used when x is in radians and that they must use radians whether stated in the question or not.

NOTES

The rest of this unit covers differentiation of more complicated functions in which the derivatives of $\sin x$ and $\cos x$ are building blocks.

8b. Differentiating exponentials and logarithms (7.2)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate functions involving e^x, $\ln x$ and related functions such as $6e^{4x}$ and $5 \ln 3x$ and sketch the graphs of these functions;
- be able to differentiate to find equations of tangents and normals to the curve.

TEACHING POINTS

It is vital that students understand the functions e^x and $\ln x$ and do not just learn how to differentiate them. Use a graphing tool to show that the gradient of a special curve $y = a^x$ has a gradient which is exactly a^x. In other words its rate of growth is exactly the same as its value at that point. This models biological growth in nature (and decay if we consider a^{-x}) The curve sits between 2^x and 3^x and has a value of 2.718… We call this exponential e.

Therefore if $y = e^x$, $\frac{dy}{dx} = e^x$.

Explain that if $y = 2e^x$ then $\frac{dy}{dx} = 2e^x$.

The students could verify this on the graphs below as Fig. 1 is effectively a stretch parallel to the y-axis.

Fig. 2 shows that the graph of $y = e^{2x}$ is twice as steep as e^x, hence if $y = e^{2x}$ then $\frac{dy}{dx} = 2e^{2x}$.

These results will be deduced more formally in Unit 8c.

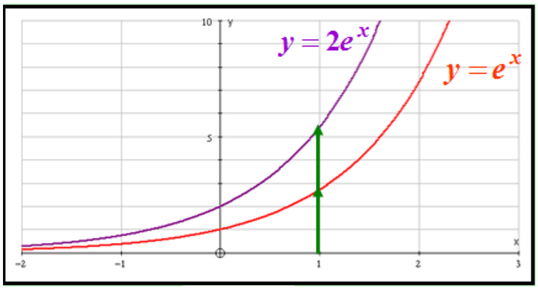


Fig. 1

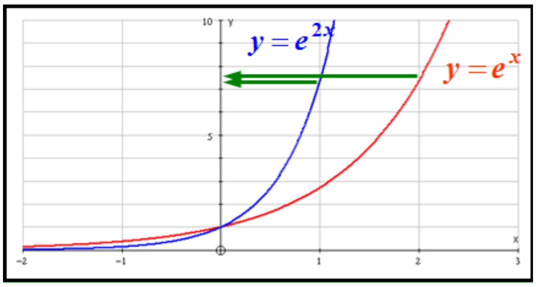


Fig. 2

For natural logarithms, recap the basic definition and graphs (from Pure Paper 1)

By looking at the graph we can see that the gradient of $y = \ln x$ at any particular point is the reciprocal of the x-coordinate of that point where the tangent is drawn. Therefore for $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$.

This can be derived in the following way:

If $y = \ln x$, then, from our definition of logs, $x = e^y$. [Write $2 = \log_{10}100$ and $100 = 10^2$ to illustrate this.]

We can differentiate $x = e^y$ by finding $\frac{dx}{dy}$ instead of the usual $\frac{dy}{dx}$.

$\frac{dx}{dy} = e^y$, and taking the reciprocal of both sides gives $\frac{dy}{dx} = \frac{1}{e^y}$.

We know that $e^y = x$ from above, so this gives $\frac{dy}{dx} = \frac{1}{x}$ as the derivative of $y = \ln x$.

The graphical approach could then be used to investigate why, for example, $y = \ln(3x)$ also has a derivative of $\frac{1}{x}$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Find gradients and normals for exponential and log functions, using graphs to check and enhance the solutions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often miss out minus signs or add an extra x into the answer when differentiating expressions like $\mathrm{e}^{-\frac{1}{4}x}$.

Some students mix up $\frac{\mathrm{d}x}{\mathrm{d}y}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ and others struggle to differentiate functions involving ln. For example given when differentiating $y = \ln 6x$ they write $\frac{1}{6x}$ rather than $\frac{1}{x}$.

NOTES

Increasingly, exam questions focus on the ability to rearrange and solve equations involving e^x and $\ln x$.

8c. Differentiating products, quotients, implicit and parametric functions. (7.2) (7.4) (7.5)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate composite functions using the chain rule;
- be able to differentiate using the product rule;
- be able to differentiate using the quotient rule;
- be able to differentiate parametric equations;
- be able to find the gradient at a given point from parametric equations;
- be able to find the equation of a tangent or normal (parametric);
- be able to use implicit differentiation to differentiate an equation involving two variables;
- be able to find the gradient of a curve using implicit differentiation;
- be able to verify a given point is stationary (implicit).

TEACHING POINTS

Most students will be able to differentiate simple instances of e^{3x}, $\sin 3x$ and $\ln 3x$ without needing formal methods such as $\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x) = \frac{f'(x)}{f(x)}$.

Many will also be able to differentiate expressions such as $(3x + 7)^5$ without using the formal method $\frac{\mathrm{d}}{\mathrm{d}x}(f(x))^n = n(f(x)^{n-1})f'(x)$.

When using the chain rule and the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$, initially u can be given to students, but they must be able to choose their own u and should move onto this quickly. Encourage students to lay work out carefully, using correct notation and $\frac{\mathrm{d}y}{\mathrm{d}u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$, not always $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Teaching should focus on *how* students know a function needs to be differentiated using the chain rule (or function of a function) and *why* a particular u is selected.

As an introduction for the product rule, ask the students to differentiate x^4. If you rewrite this as the product $(x^2)(x^2)$ and differentiate each part separately, it does not match $4x^3$. Using the product rule will give that match.

In a similar way, writing x^4 as $\frac{x^3}{x}$ can lead into the quotient rule.

Work involving the product and quotient rule often breaks down because of weak algebraic skills and this needs plenty of practice. Students should practice fully simplifying their answers as they may be asked to give a solution in a particular form. Encourage students to lay work out carefully. Good notation is vital to achieve success.

Show that the product rule and the quotient rule give the same answers on functions that can be written in two ways, for example, $y = \frac{x+1}{x+2}$ and $y = (x + 1)(x + 2)^{-1}$.

Also show that the chain rule and the product rule give the same derivative for $\cos^2 x$ and $\sin^2 x$.

Use the product and quotient rules to derive the differentials of some key trigonometric expressions. For example $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)$ using the quotient rule giving $\sec^2 x$.

For parametric differentiation, make links with the chain rule to give $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$

Stress that we often substitute in the value of the parameter t at the point which we need to find the gradient

Many questions will involve trigonometric functions, so students must be fluent at differentiating these.

For implicit differentiation, consider the equation of a circle, $x^2 + y^2 = 16$. To differentiate this function we would have to make y the subject of the formula. Sometimes this can be difficult or even impossible.

Make sure students can confidently differentiate terms like x^2y using implicit differentiation. Finally, stress that we need to substitute in *both* x and y coordinates to find the gradient at a certain point.

Students may have to apply the product or quotient rules in implicit differentiation questions and should be given examples of this. In exam questions students are almost always required to find the gradient through implicit differentiation.

Take a point on a circle or another type of curve and find the gradient using two both parametric and implicit differentiation. Then find the equation of tangent and/or normal and see that both methods give the same answer.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Although repeated chain rule questions rarely appear in the exam they provide good extension material and provide an excellent test of good method and correct mathematical notation. Extend the students further by asking them to look up a proof or derivation of the product and quotient rules.

Use the methods above to work out the derivative of a general exponential function, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = ka^{kx}\ln a$

Give the students lots of mixed questions which will enable them to select the correct method. Discussion should focus on why they have selected a particular method and quick ways of identifying the correct method.

Students must be able to use all methods as a particular method is sometime specified in the exam.

Some questions require a trigonometric identity in order to simplify the solution.

The specification states that ‘differentiation of arcsin$x$, arccos$x$, and arctan$x$ are required’.

Cover questions involving finding tangents, turning points and normals (this links with Unit 6d).

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors involve: not using the method specified; algebraic errors when manipulating expressions; and being unable to identify the need of the product rule and instead simply differentiating the separate parts and multiplying.

NOTES

Check which differentials are in the formula book and which must be learnt.

This work links to Unit 11f (Differential equations), after more integration skills have been developed.

8d. Second derivatives (rates of change of gradient, inflections) (7.1b)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find and identify the nature of stationary points and understand rates of change of gradient.

TEACHING POINTS

The specification states 'Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection' and 'know that at an inflection point $f''(x)$ changes sign.'

The basic principle is usually

	$\frac{dy}{dx}$ or $f'(x)$	$\frac{d^2y}{dx^2}$ or $f''(x)$
maximum	$= 0$	< 0
minimum	$= 0$	> 0

However show examples of curves in which $\frac{d^2y}{dx^2}$ or $f''(x) = 0$, where there could beva point of inflexion (or not). i.e. The rate of change of gradient is zero.

We would need to work out $f'(x)$ and scrutinise gradient either side of the point x. There may be positive or negative inflexion or neither (depending on the nature of the curve, which could be convex or concave).

Use graph drawing packages to investigate the shapes and turning points of various curves of the type $y = ax^n$ $(n > 2)$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at $y = \frac{a}{x}$, e^x, $\ln x$ etc.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students should be encouraged to state "$\frac{dx}{dy} = \ldots$ when $x = \ldots$", especially when finding a given answer.

An easy mistake students may make is to mix up maxima and minima.

NOTES

The types of functions and complexity of expressions are consistent with the functions covered earlier in this unit.

8e. Rates of change problems* (including growth and decay) (7.6)	**Teaching time**
***see also Integration (part 2) – Differential equations**	3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use a model to find the value after a given time;
- be able to set up and use logarithms to solve an equation for an exponential growth or decay problem;
- be able to use logarithms to find the base of an exponential;
- know how to model the growth or decay of 2D and 3D objects using connected rates of change;
- be able to set up a differential equation using given information which may include direct proportion.

TEACHING POINTS

This content links to kinematics, where velocity is considered as $\frac{\mathrm{d}s}{\mathrm{d}t}$ and acceleration as $\frac{\mathrm{d}v}{\mathrm{d}t}$.

The example below is from the original SAMs:

A team of conservationists is studying the population of meerkats on a nature reserve.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{22}P(11 - 2P), \quad t \geq 0$$

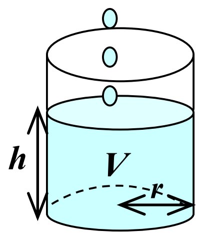
where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that there are 1000 meerkats on the nature reserve when the study began,

(a) determine the time taken, in years, for this population of meerkats to double,

(b) show that the population cannot exceed 5500.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING



Consider water entering this cylinder. To work out the rate at which the height is increasing we need to calculate $\frac{\mathrm{d}h}{\mathrm{d}t}$.

In exam questions, the rate that the volume of water increases at is often given as $\frac{\mathrm{d}V}{\mathrm{d}t}$.

Therefore, we need to use the chain rule to create $\frac{\mathrm{d}h}{\mathrm{d}t}$ from $\frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V}$ so we need a formula connecting h and V.

$V = \pi r^2 h$ and from this we can work out $\frac{\mathrm{d}V}{\mathrm{d}h}$ and then $\frac{\mathrm{d}h}{\mathrm{d}V}$ etc.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Most students are able to substitute correctly into a formula for exponential growth and decay.

When required to set up an inequality most students showed that they understood the information given and wrote down a correct opening expression, although there was uncertainty over which way the inequality should go. Some then simplified and solved using logarithms efficiently to get the correct answer. Some resorted to trial and improvement which was accepted for full marks if done correctly, but was worth no marks otherwise.

When solving equations involving exponentials, knowledge of using logarithms varied widely. Many were unable to deal properly with the coefficient and the exponential term and wrote down equations in which t actually should have cancelled out.

Some care needs to be taken when interpreting the answers to exponential growth and decay questions to ensure they are given in the correct form e.g. to the nearest year, second etc.

NOTES

For first order differential equations (which require separating variables) see Unit 11 – Integration (part 2)

UNIT 9: Numerical methods

Return to overview

SPECIFICATION REFERENCES

9.1 Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of x on which $f(x)$ is sufficiently well-behaved
Understand how change of sign methods can fail

9.2 Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams
Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = g(x_n)$
Understand how such methods can fail

9.4 Use numerical methods to solve problems in context

PRIOR KNOWLEDGE

Covered so far

- Series, sequences and recurrence relations (Unit 4)
- Graphs, roots and functions
- Differentiation

GCSE (9-1) in Mathematics at Higher Tier

A15, A20 Iterations and approximate areas under curves
A15 Kinematics (velocity–time graphs)

AS Mathematics – Pure Mathematics content

2.7, 2.8 Graphs, roots and functions
7, 8 Differentiation and integration (See Units 6 &7 of SoW)

AS Mathematics – Mechanics content

7.2 Kinematics (velocity–time graphs) (See Unit 7 of SoW)

KEYWORDS

Roots, continuous, function, positive, negative, converge, diverge, interval, derivative, tangent, chord, iteration, Newton-Raphson, staircase, cobweb, trapezium rule.

NOTES

This topic extends the work done on iterations at GCSE (9-1) Mathematics and also links with graphs and functions.

9a. Location of roots (9.1)	**Teaching time** 1 hour

OBJECTIVES

By the end of the sub-unit, students should:

- be able to locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$;
- be able to use numerical methods to find solutions of equations.

TEACHING POINTS

Students should be able to recognise that a root exists when there is a change of sign of $f(x)$. Students should recognise this and remember it. There is often an easy mark missed on the exam for this because it is phrased slightly differently.

Students should know that sign change is appropriate for continuous functions in a small interval.

When the interval is too large the sign may not change as there may be an even number of roots.

If the function is not continuous, the sign may change but there may be an asymptote (not a root) so the method will fail.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at continuous functions and then contrast this with say $y = \frac{1}{x}$ and $y = \tan x$, which will not have any roots in some intervals despite a change of sign. Use graph drawing packages to investigate similar behaviour in other functions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students must define $f(x)$ before substituting x-values to find a root.

Most students can successfully identify the root of equations. However there are still many students who then write “change of sign therefore a root” without clarification of where the root lies and hence loose a mark.

Marks are sometimes lost unnecessarily if students do not give their answers to the specified number of significant figures or decimal places.

NOTES

Iterations may be suggested for solving equations which cannot be solved by analytic means (see the next section).

9b. Solving by iterative methods (knowledge of 'staircase and cobweb' diagrams) (9.2)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the principle of iteration;
- appreciate the need for convergence in iteration;
- be able to use iteration to find terms in a sequence;
- be able to sketch cobweb and staircase diagrams;
- be able to use cobweb and staircase diagrams to demonstrate convergence or divergence for equations of the form $x = \mathrm{g}(x)$.

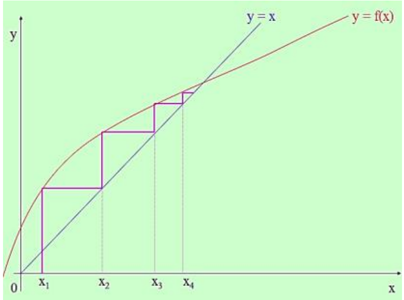
TEACHING POINTS

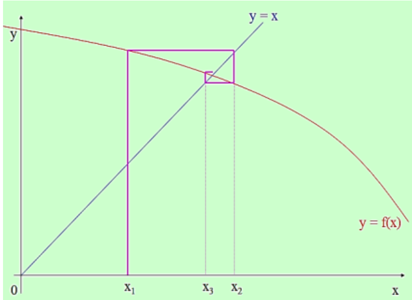
Students will have met iterations at GCSE (9-1) Mathematics, but will need to be introduced to some of the conditions for convergence and understand how the process works (and sometimes does not work).

Revise the method to make one of the x's the subject of the formula, leading to $x = \mathrm{f}(x)$. Use graph-drawing packages to look at the function and decide where would be appropriate for the first iteration value (i.e. x_0).

The method at A level is to consider the roots of the function $y = \mathrm{f}(x)$ as the intersection of the two functions $y = x$ and $y = \mathrm{f}(x)$ (hence $x = \mathrm{f}(x)$).

Use an iteration of the form $x_{n+1} = \mathrm{f}(x_n)$ to find a root of the equation $x = \mathrm{f}(x)$ and show how the convergence can be understood in geometrical terms by drawing cobweb and staircase diagrams like those shown here.



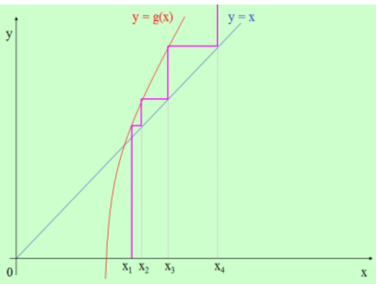


OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Which iterations converge or diverge?

Are there any values which cannot be substituted into certain iterations?

Why does this staircase diagram fail?



COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks will be lost due to using degrees (instead of radians) if functions involve trigonometric terms.

Choosing an unsuitable interval will also prevent progress in these questions.

NOTES

Students should understand that many mathematical problems cannot be solved analytically, but that numerical methods permit a solution to be found to a required level of accuracy.

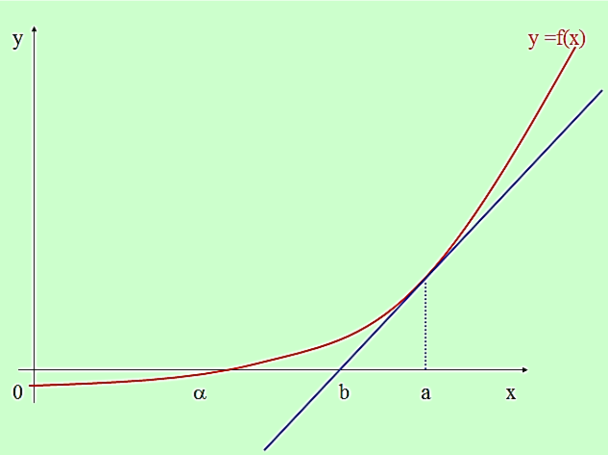
9c. Newton-Raphson method (9.2)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve equations approximately using the Newton-Raphson method;
- understand how the Newton-Raphson method works in geometrical terms.

TEACHING POINTS

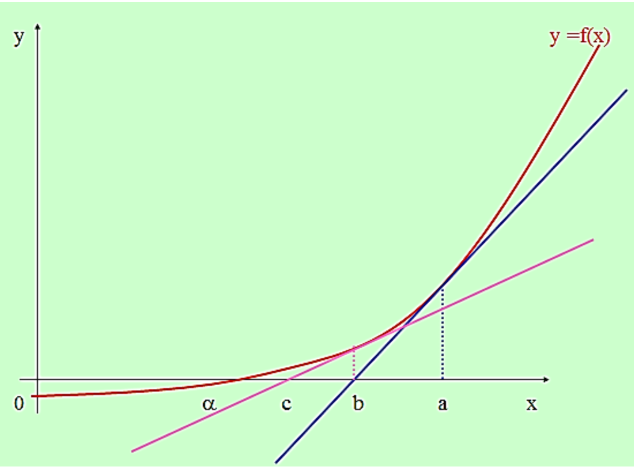


Consider the diagram above. The tangent crosses the x-axis at b (which is quite near the actual root α).

By considering the gradient of the tangent, we get $f'(a) = \frac{f(a)}{a-b}$ which can be rearranged to give $b = a - \frac{f(a)}{f'(a)}$.

We therefore have an expression for an approximation of the root (b), which uses the equation of the curve and its derivative at the point a.

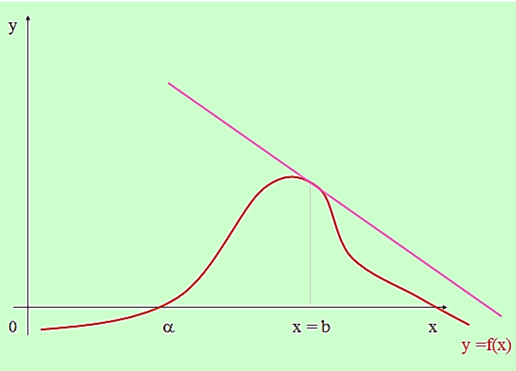
If we now go up from the point b, hit the curve and then construct another tangent (as in the diagram below) then, a similar argument, gives a better approximate root at c (nearer than b). Therefore we would get $c = b - \frac{f(a)}{f'(a)}$.



So if we continued this process we would get $d = c - \frac{f(a)}{f'(a)}$ and generally $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Sometimes the process fails for some curves or starting points.

What happens to the tangent if we try to apply the process here?



An example of the type question which may be seen:

> f $(x) = x^3 + 8x - 19$.
>
> Obtain an approximation to the real root of f$(x) = 0$ by performing two applications of the Newton-Raphson procedure to f(x), using $x = 2$ as the first approximation.
>
> Give your answer to 3 decimal places.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Try different methods to find the roots of the same function. Which is the most efficient method or leads to the more accurate approximation? Consider, for example, iteration vs Newton–Raphson.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks are often lost for sign errors and other numerical slips.

Students must show full working leading to the correct answer for full marks. Giving a correct answer either without working or following wrong working will result in zero marks.

NOTES

Graph drawing packages are an essential way to ‘look’ at the curve and the potential position of the roots depending on the first approximation of the root.

There will be a rich source of questions from the legacy FP1 papers as this topic was part of that specification.

Functions used will be consistent with the differentiation unit, e.g. e^{2x}, etc.

9d. Problem solving (9.4)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use numerical methods to solve problems in context.

TEACHING POINTS

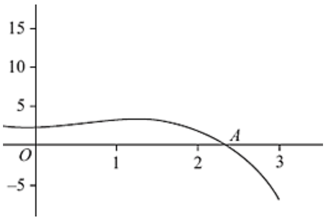
Recurrence relations, iterations and Newton-Raphson methods can be used obtain approximate solution(s) to an equation set in a context. The important point to make is that the original equation is too difficult to solve algebraically (e.g. the roots are decimal and/or the functions will not factorise or contain terms which are non-polynomials).

The choice of degree of accuracy is dependent upon the context of the problem, e.g. nearest minute or number of years.

An example of a possible question is as follows.

The equation $P = -t^3 + 2t^2 + 2$ $(t > 0)$ represents a share price p, at time t months after the money was invested.

The iteration $t_{n+1} = \frac{2}{(t_n)^2} + 2$ represents the solution to the above equation.



Taking $t_0 = 2.5$ months, show that the root gives an approximation to when the share price has zero value.
Use the iteration to find the (converged) time at which the shares lose their value before going negative.
When were the shares at their highest value?

Can Newton-Raphson be used to find the approximate solution of the above relationship?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Which approximation method (when a choice is possible) gives the most efficient solution?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Questions in context (other than the trapezium rule) are not in the legacy specs so no examination data is available.

NOTES

The specification states: ‘iterations may be suggested for the solution of equations not soluble by analytic means’.

For approximate areas under curves to find the displacement (distance travelled) under a velocity (speed)-time graph, see Unit 11e – Trapezium rule.

UNIT 10: Integration (part 1)

Return to overview

SPECIFICATION REFERENCES

8.2 Integrate x^n, (including $\frac{1}{x}$) and integrate e^{kx}, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples

To include integration of standard functions such as $\sin 3x$, $\sec^2 2x$, $\tan x$, e^{5x}, $\frac{1}{2x}$.

Students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$.

8.5 Students should recognise integrals of the form $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$.

PRIOR KNOWLEDGE

Covered so far

- Knowledge of e^x and $\ln x$
- Laws of logarithms
- Trigonometry
- Differentiation

AS Mathematics – Pure Mathematics content

6.1, 6.3 Knowledge of e^x and $\ln x$ (See Unit 8 of SoW)

6.4 Laws of logarithms (See Unit 8 of SoW)

5.1 Trigonometry (See Unit 4 of SoW)

7, 8 Differentiation and integration (See Units 6 &7 of SoW)

KEYWORDS

Integral, inverse, differential, coefficient, index, power, negative, reciprocal, natural logarithm, $\ln |x|$, coefficient, exponential, identity, sin, cos, tan, sec, cosec, cot, e^x.

NOTES

This first part of Integration is about using the reverse process of differentiation and applying previously leant skills. The next part will use further techniques for the integration of combined functions as well as looking at applications of integration.

10a. Integrating x^n (including when $n = –1$), exponentials and trigonometric functions (8.2)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate expressions by inspection using the reverse of differentiation;
- be able to integrate x^n for all values of n and understand that the integral of $\frac{1}{x}$ is $\ln |x|$;
- be able to integrate expressions by inspection using the reverse of the chain rule (or function of a function);
- be able to integrate trigonometric expressions;
- be able to integrate expressions involving e^x.

TEACHING POINTS

Recap all the methods of differentiation covered earlier in the course. This can also be used as a starting point for introducing the different rules for integration.

Consider the integral of $x^{-1} = \frac{1}{x}$. Using the rule from AS Mathematics – Pure Mathematics gives. $\frac{1}{0}$. However, if we recall that the differential of $\ln |x|$ is $\frac{1}{x}$, then the reverse operation tells us that the integral of $\frac{1}{x}$ is $\ln |x| + c$. Similarly, the differential of e^x is e^x, so the integral will also give the same result.

Finally, the differential of trig expressions should be recapped as this also leads to some standard results for trigonometric integrals.

Take care to show how the integral of $\sin x$ is $–\cos x + c$ (as the differential of $\cos x$ leads to $–\sin x$).

The integral of $\sec^2 x$ looks difficult but is only the reverse of the differential of $\tan x$.

Students must end all indefinite integrations with $+ c$ and use correct notation when integrating and must include dx.

Encourage students to develop their own technique for integrating problems which require the reverse chain rule. If good examples are used, most students will be able to work out their own method and soon be able to write down the answers directly for integrals like $3e^{2x}$ and $4 \sin (3x)$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

It is always a good idea to advise students to differentiate their answer to see if it goes back to the original expression (pre-integration). This is a good way to check for sign errors, particularly with the trigonometric questions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

In exam situations, many students incorrectly integrate functions involving e^x by dividing by the x.

Algebraic errors are also fairly common; clear working and good notation can help here.

NOTES

Make sure students are fluent in these basic integrals, as this will increase the likelihood of success with the remainder of this unit.

10b. Using the reverse of differentiation and using trigonometric identities to manipulate integrals (8.2)	Teaching time 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- recognise integrals of the form $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$;
- be able to use trigonometric identities to manipulate and simplify expressions to a form which can be integrated directly.

TEACHING POINTS

Consider the rule for differentiating ln $|f(x)|$. This was $\frac{f'(x)}{f(x)}$. A special case of this is the integral of $\frac{1}{x}$, which is ln $|x|$ (+ c).

So, if we have to integrate an expression in which the top of the fraction is the exact differential of the denominator (or a multiple of it), then the answer is the natural log of the denominator (+ c).

Make sure students can adjust questions like the integral of $\frac{4x^2}{x^3}$.

Consider examples like the integral of $\tan x$ by rewriting it as $\frac{\sin x}{\cos x}$, leading to a natural log answer (be careful with the minus!)

One of the most common integrals is $\cos^2 x$. The standard method for integrating this is to rearrange the appropriate double angle formula to create an integral involving not x^2 but $2x$ which is much easier to directly integrate (as shown in the previous section).

Students will need lots of practice in selecting the correct version of $\cos 2x$, which involves only $\cos^2 x$ terms and then rearranging it.

The specification states: 'students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$'.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students must have lots of practice at working with logarithms and exponentials when integrating, and should leave their answers in exact form. They also need to be fluent in knowing the key trig identities and how to manipulate them from the ones in the formula book.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors seen include: mistakes when arranging and substituting identities into integrals; and incorrectly applying laws of logarithms.

NOTES

Log integrals are vital when working with the partial fractions and many of the differential equations in the next unit.

UNIT 11: Integration (part 2)

Return to overview

SPECIFICATION REFERENCES

8.3 Use a definite integral to find the area under a curve and the area between two curves

8.4 Understand and use integration as the limit of a sum

8.5 Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively

8.6 Integrate using partial fractions that are linear in the denominator

8.7 Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions

8.8 Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics

9.3 Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between

PRIOR KNOWLEDGE

Covered so far

- Laws of logarithms
- Trigonometry
- Partial fractions
- Differentiation

GCSE (9-1) in Mathematics at Higher Tier

A15 Areas under curves

AS Mathematics – Pure Mathematics content

7, 8 Differentiation and integration (See Units 6 &7 of SoW)

6.4 Laws of logarithms (See Unit 8 of SoW)

5.1 Trigonometry (See Unit 4 of SoW)

AS Mathematics – Mechanics content

7.2 Kinematics (velocity–time graphs) (See Unit 7 of SoW)

KEYWORDS

Integral, definite integral, integrand, limit, indefinite integral, constant of integration, trapezium, substitution, by parts, area, differential equation, first order, separating variables, initial conditions, general solution.

NOTES

This section completes the calculus for this course. It is also the pre-requisite for the calculus in some of the Further Mathematics units.

11a. Integration by substitution (8.5)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate expressions using an appropriate substitution;
- be able to select the correct substitution and justify their choices.

TEACHING POINTS

Most students find integration by substitution challenging and will need to complete lots of different styles of questions. It is a good idea to start with an example which can be performed by inspection as the reverse of differentiation.

Students also like to have a step by step process.

1. Use the given substitution or decide on your own. The substitution is usually the contents of a bracket, square root or the 'nasty' bit! i.e. Let $u = \ldots$
2. Differentiate the substitution i.e. $\frac{\mathrm{d}u}{\mathrm{d}x} = \ldots$
3. Make dx the subject of the formula
4. Replace the dx and make the substitution into the integrand
5. Cancel out any remaining x*
6. Integrate the resulting (simpler) integral
7. Substitute back to get the answer in terms of x again

*If there are any remaining x, you can re-use the substitution making the x the subject

For expressions including trigonometric functions, the identities involving $\sin^2 x$, $\sec^2 x$ are often useful to simplify the integrand.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Try to encourage students to experiment with different substitutions, particularly types involving expressions such as $\sqrt{3x+4}$. Do we use $u^2 = 3x + 4$ or $u = 3x+4$? The former will require implicit differentiation.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Mistakes students make when attempting to integrate by substitution include not changing the dx correctly and simply writing it as du, and failing to substitute back to give an expression in x at the end.

NOTES

Return to this method when covering areas under curves as the limits need to be changed by substituting them into the required substitution.

11b. Integration by parts (8.5)	**Teaching time** 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate an expression using integration by parts;
- be able to select the correct method for integration and justify their choices.

TEACHING POINTS

It is a good idea to show how the product rule for differentiation can be integrated on both sides to derive the 'by parts' formula (which is given in the formulae booklet).

Students are usually able to start questions using this method but struggle to get to full solutions and will require lots of practice with algebraic manipulation.

Time should be spent discussing the choice of u and dv. It is usually advisable to select the polynomial to be the u as it simplifies to a lower power after calculating du, thus making the second integral easier than the original question.

Students should recognise that $\ln x$ cannot be integrated simply and should therefore always be chosen as u.

$\ln x$ itself can be integrated using this method taking $u = \ln x$ and dv = 1 (as we cannot integrate $\ln x$, but can differentiate it to give $\frac{1}{x}$). The dv becomes more complicated, but then simplifies in the second integral with the $\frac{1}{x}$.

More able students should be able to access questions where it is necessary to use integration by parts twice (e.g. $u = x^2$).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider the integral of $e^x \cos x$ and show that the application of 'by parts' loops back to the original question. Refer to the equation $x = 4 - x$ and contrast this with the structure of this example.

Let the original question be I (for integral) and this can lead to $2I = \ldots$.

[This is a pre-requisite for reduction formulae in Further Pure Mathematics.]

Students should integrate functions such as $\int x(x+3)^6 \mathrm{d}x$ using both 'by parts' and 'substitution' to show that they give the same answer. This is a good activity for discussion as initially they appear to be different, but after some algebraic manipulation give the same answer.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors when integrating by parts include: choosing u and dv incorrectly (in particular $\ln x$ must always be chosen as u); algebraic errors – especially if they do not remove any common factors to outside the integral sign; incorrect coefficients when integrating dv; and sign errors where sin and cos are involved.

NOTES

The method of integration by parts may be specified in the question.

Revisit this method when finding areas under curves (introducing limits) and/or the trapezium rule (for approximate areas).

11c. Use of partial fractions (8.6)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate rational expressions by using partial fractions that are linear in the denominator;
- be able to simplify the expression using laws of logarithms.

TEACHING POINTS

Revise the simplification of rational expressions into partial fractions. We have already seen that this technique is useful in binomial expansions.

Often the first part of an integration question of this sort will ask students to split the fraction into two (or more) partial fractions.

The next part will then ask for the integration to be carried out. For example:

Integrate $\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x$.

This will lead to $\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x = \int \left(\frac{1}{x-1} - \frac{3}{3x+2}\right)\mathrm{d}x = \ln(x-1) - \ln(3x+2)\ (+\ c)$

It is sometimes sufficient to leave the answer in this form, but 'Show that' questions will influence the further simplification using laws of logs.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Although the specification states 'linear in the denominator', you may want to cover repeated factors, which will lead to, for example, $(x-2)^2$ in the denominator, which will not be a log integral.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Partial fractions questions are generally done well though some students attempt to integrate the numerator and denominator separately without using partial fractions.

NOTES

These integrals will sometimes be tested via a differential equation later in the course and laws of logs will form a vital role in finding the general solution. Definite integrals may also need to be calculated and simplified numerically. e.g. $\ln 6 - \ln 2 = \ln\frac{6}{2} = \ln 3$.

11d. Areas under graphs or between two curves, including understanding the area is the limit of a sum (using sigma notation) (8.3) (8.4)

Teaching time
4 hours

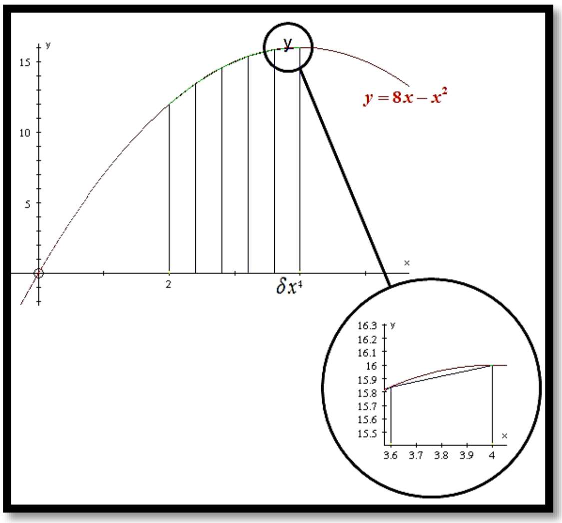
OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use integration as the limit of a sum;
- understand the difference between an indefinite and definite integral and why we do not need + c;
- be able to integrate polynomials and other functions to find definite integrals, and use these to find the areas of regions bounded by curves and/or lines;
- be able to use a definite integral to find the area under a curve and the area between two curves.

TEACHING POINTS

Begin by showing a sketch of the curve and spit the area below it into thin strips, as shown below.



Now each strip is of elemental width δx, so the approximate area of each strip is $y\delta x$, where y is the height of each strip measured on the y-axis. If we sum all the strips, this would give us the total area below the curve. If the first strip starts at the point (2, 0) and the last strip ends at (4, 0), these become the limits on the definite integral. We can think of '4' as the area up to 4 and '2' as the area up to 2 (both measured across from the y-axis or x = 0).

We have seen from work on series, that we can use the sigma notation for sums so we can represent the area as $\sum y\delta x$. As δx gets thinner and thinner, the area becomes more accurate as the strips become more like rectangles. (This links nicely with the trapezium rule in the next sub-unit.)

We say that 'in the limit, as δx approaches zero' the sum becomes continuous rather than discrete and we can replace y with f(x) and $y\delta x$ becomes f(x)δx.

It happens that the rule for integration (which so far has only been used as the reverse of differentiation) gives the exact area under the curve. We can substitute in a and b, where the area's strips began and ended, as the limits of integration. The $y\delta x$ becomes f(x)δx and for the integral becomes f(x)dx. In other words the δx is the dx we have always understood as 'with respect to x'.

This leads to, $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$

Do lots of work on finding areas that require more than just a simple integral to be evaluated, for example when some of the area is below the x-axis or when finding the area between a line and a curve.

For example:

Find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$.

Find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.

Encourage students to always do a sketch or use a graph drawer to help with such questions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider questions which have part of the graph below the x-axis, in which the area is negative. This time the roots are vital as we have to create two separate regions to calculate the total area. Show that just integrating between the start and end points will give a wrong result as the areas will subtract form each other.

Sometimes, you can create a new equation by subtracting the two areas *before* you integrate (when you have two curves and have to find the area between them):

$$\int_a^b y_1\,dx - \int_a^b y_2\,dx = \int_a^b (y_1 - y_2)\,dx$$

Care is needed with this method, and you should emphasise to students that they need to sketch it first making sure y_1 is higher than y_2.

Include questions where the area is found between a curve and the y-axis using $\int x\,dy$, with y-coordinates as limits.

Finally, consider areas which are bounded by curves defined by other types of functions, e.g. $y = e^{2x}$ or $y = \ln x$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The method for answering these types of exam questions is often understood, but many students loose accuracy marks due to arithmetical errors or using incorrect limits.

NOTES

Link this section to the trapezium rule which follows next.

11e. The trapezium rule (9.3)	**Teaching time** 2 hours

OBJECTIVES

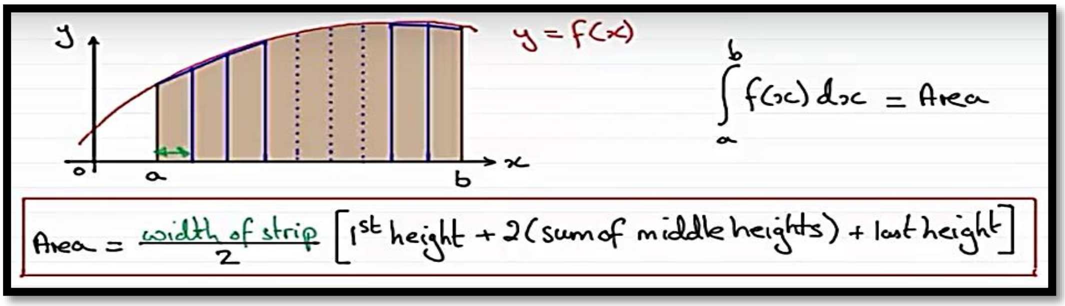
By the end of the sub-unit, students should:

- be able to use the trapezium rule to find an approximation to the area under a curve;
- appreciate the trapezium rule is an approximation and realise when it gives an overestimate or underestimate.

TEACHING POINTS

Make a direct link with the previous section and how to find an estimate for the area under a curve by dividing it into a finite number of strips. Sometimes an estimate is all that we need, and sometimes the integral is very complicated (or sometimes impossible) to integrate and so we have to estimate the area numerically.

The trapezium rule is given in the formula book (and may have also been covered in GCSE (9-1)). Students who struggle with algebra sometimes prefer to use the word version below:



Some students may be able to derive the rule by adding all the individual strips areas (i.e. $\frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \ldots$) and then factorising to give the trapezium rule as in the formula book.

Ask students to calculate $\int_0^1 x\mathrm{e}^{2x}$ by using integration by parts and also by completing the table and using the trapezium rule (this is the quicker method). They should compare the answers they get using the different methods.

x	0	0.2	0.4	0.6	0.8	1
$y = x\mathrm{e}^{2x}$	0	0.29836		1.99207		7.38906

Another example of the type of question that may be asked is:

Evaluate $\int_0^1 \sqrt{2x+1}\,\mathrm{d}x$ using the values of $\sqrt{2x+1}$ at x = 0, 0.25, 0.5, 0.75 and 1.

Make a sketch of the graph to determine whether the trapezium rule gives an over-estimate or an under-estimate of the exact value of the integral.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The following exam question shows a modelling example:

A river, running between parallel banks, is 20 m wide. The depth, y metres, of the river, measured at a point x metres from one bank, is given by the formula:

$y = \frac{1}{10}x\sqrt{20 - x}$, $0 \leq x \leq 20$

(*a*) Complete the table below, giving values of y to 3 decimal places.

x	0	4	8	12	16	20
y	0		2.771			0

(*b*) Use the trapezium rule with all the values in the table to estimate the cross-sectional area of the river.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When using the trapezium rule students sometimes mix up the number of strips and the number of x or y values.

The other main place marks are lost is not giving the final answer to three significant figures.

NOTES

Make sure that you use the same form for the trapezium rule as that given in the formula book.

11f. Differential equations (including knowledge of the family of solution curves) (8.7)(8.8)	**Teaching time** 4 hours

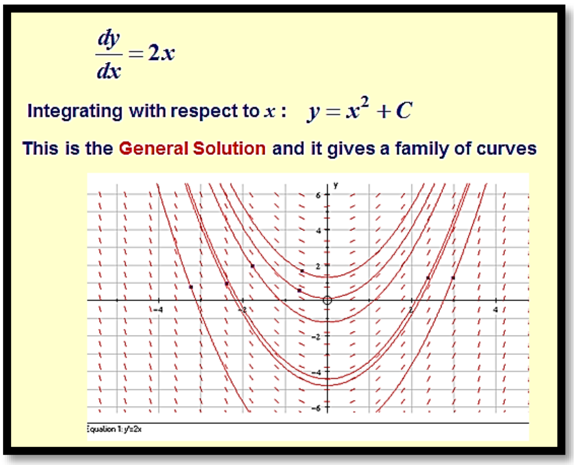
OBJECTIVES

By the end of the sub-unit, students should:

- be able to write a differential equation from a worded problem;
- be able to use a differential equation as a model to solve a problem;
- be able to solve a differential equation;
- be able to substitute the initial conditions or otherwise into the equation to find + *c* and the general solution.

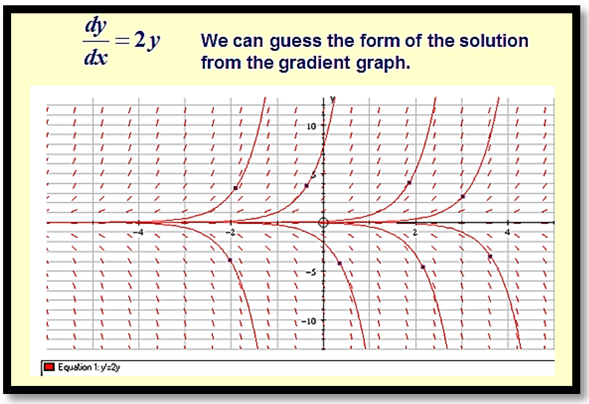
TEACHING POINTS

Begin by considering the simplest possible differential equation (defined as first order) as below.



Notice that the graph drawing tool can plot the differential equation to give a family of curves which mirror the solution (family of parabolas)

The next differential equation is more difficult as we cannot integrate directly because the variable is *y* rather than *x*. But looking at the family of curves may give us a clue about the eventual solution.



The curves look like exponentials.

The solution can be performed by using a method called 'separating variables', in which we rearrange and split up the $\frac{\mathrm{d}y}{\mathrm{d}x}$ as if it is a fraction. It is vital to keep all the y's and dy's *and* the x's and dx's together, but also the dx and dy must be in the numerator on each side.

The full solution is shown below.

$$\frac{dy}{dx} = 2y$$

To find the general solution by Calculus, we need to integrate with respect to x, yet the variable on the r.h.s. is y

Separating the variables, we get

$$\frac{dy}{y} = 2dx$$

Notice that the operator is separated.

The constant 2 can be on either side as can the +C.

We can now integrate: $\int \frac{dy}{y} = \int 2\ dx \quad \Rightarrow \quad \ln y = 2x + C$

This can be written as: $y = e^{2x+C}$

Index laws can split e^{2x+C}

Also, e^{C} = A (another constant)

which can be simplified to $y = e^{2x}e^{C}$ or $y = Ae^{2x}$

$$\boxed{\frac{dy}{dx} = ky \quad \Rightarrow \quad y = Ae^{kx} \quad \text{where } k \text{ is a constant}}$$

As suspected, the family of curves were exponential curves.

$y = Ae^{2x}$ is a general solution, but how do we find the value of the constant A? We need to have some information about the data from which the differential equation originates. Something along the lines of 'when $x = 0$, $y = 2$'.

Substituting this pair of values into the general solution and finding the value of A, will lead to a particular solution.

Sometimes we may have a choice of pairs to substitute or we may have two pairs of values in order to work out two constants.

Explain that questions may be set in a context and, in these cases, students need to interpret the solution of the differential equation in the context of the problem. This may including identifying limitations of the solution.

The following example is typical:-

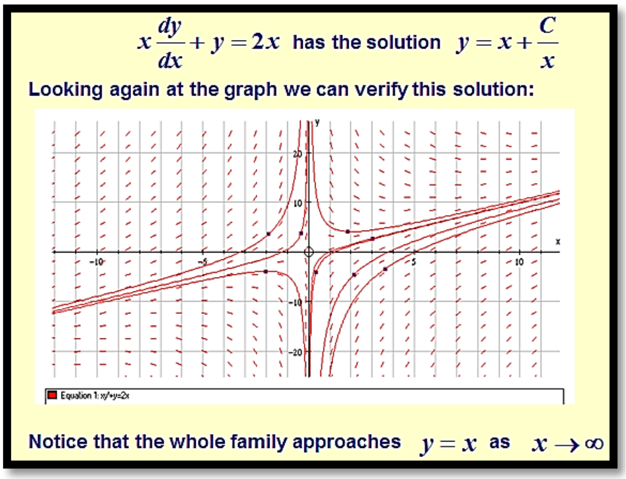
> The population of a town was 50 000 in 2010 and had increased to 55 000 by 2015. Assuming that the population is increasing at a rate proportional to its size at any time, estimate the population in 2020 giving your answer to the nearest hundred.

$\frac{db}{dt} = kn \quad \Rightarrow \quad n = Ae^{kt}$ as above, but now n is the number of people and t is the time in years.

The validity of the solution for large values should be considered, for example, if the question was modelling population growth; would it be realistic for the value to keep increasing forever?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The example below has a family of curves which has elements of both $y = x$ and $y = \frac{1}{x}$, but it seems that the $y = x$ is trying to win!



Also, for separating variables and finding the particular solution, encourage the more able students to use the initial conditions as the limits of integration, thus avoiding the + c.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiner comments indicate that this can prove a difficult topic for some students:

When forming a differential equation some students wrote down the correct differential equation apparently fully understanding all the information given and interpreting it correctly. However, all sorts of errors abounded in other attempts, some not even involving a derivative, and some with derivatives in x and y.

Many had a spurious t and/or h, either as a multiple or power, and the k appeared in a variety of places. Some students did not even form an equation, leaving a proportionality sign in their answer.

When solving a differential equation most students knew they were expected to separate the variables and did it correctly, although there were some notation errors in the positioning of dx, at the front rather than the rear of the integrand. Those who failed to separate the variables, just produced nonsense. Many students struggled with the fact that integration by parts or substitution was needed. All students, no matter what their attempt at the integral, could obtain a method mark if they included a constant and tried to find it using the given initial conditions.

NOTES

Link this topic to kinematics. For example solving differential equations of the form $\frac{\mathrm{d}v}{\mathrm{d}t} = 3t^2$ (when $t = 0$, $v = 4$). Separating variables leads to $v = t^3 + c$ etc.

$\frac{\mathrm{d}v}{\mathrm{d}t}$ is the acceleration and this shows that if we integrate the acceleration, we get the velocity.

UNIT 12: Vectors (3D) **Use of vectors in three dimensions; knowledge of column vectors and i, j and k unit vectors (10.1)**	**Teaching time** 5 hours

Return to overview

SPECIFICATION REFERENCES

10.1 Use vectors in three dimensions
Knowledge of column vectors and **i**, **j** and **k** unit vectors in three dimensions

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

G24, G25 Vectors

AS Mathematics – Pure Mathematics content

10 Vectors (See Unit 5 of SoW)

KEYWORDS

Vector, scalar, column, 3D coordinates, vertices, Cartesian, **i**, **j**, **k**, magnitude, origin, distance, direction, angle, position vector, unit vector, orthogonal, vector addition/subtraction.

NOTES

This topic is a natural extension of the vector work in AS Pure Mathematics. It extends to 3 dimensions via an additional vector **k**, or a third entry in the column vector.

OBJECTIVES

By the end of the sub-unit, students should:

- be able to extend the work on vectors from AS Pure Mathematics to 3D with column vectors and with the use of **i**, **j** and **k** unit vectors;
- be able to calculate the magnitude of a 3D vector;
- know the definition of a unit vector in 3D;
- be able to add 3D vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations;
- understand and use position vectors, and calculate the distance between two 3D points represented by position vectors;
- be able to use vectors to solve problems in pure mathematics and in contexts (e.g. mechanics).

TEACHING POINTS

Begin by showing some 3D coordinates on x, y, z axes. (Graph drawing packages are very useful here, especially if you can turn the grid to view from different positions.)

Consider a cuboid (2 by 3 by 4), with one corner at the origin. Ask the class to write down the coordinates of all the vertices.

Remind students of 2D work and extend to 3D column vectors, orthogonal unit vectors **i**, **j**, **k** and position vectors.

Write all the vectors from the 'origin' corner of the cuboid as position vectors (e.g. OA = $\underline{\mathbf{a}}$, etc.)

Calculate the magnitude of these vectors as $\sqrt{2^2+3^2+4^2}$ for example.

Extend this idea to calculating the distance d between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

Extend the ideas of vector addition and subtraction to 3D: $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Cover the triangle and parallelogram laws of addition, as well as demonstrating parallel vectors.

Show how to find a unit vector in the direction of $\underline{a}$, and make sure students are familiar with the notation $|\boldsymbol{a}|$ (extended to 3D).

Use vectors to solve problems in pure mathematics and discuss the 3D geometrical interpretations of solutions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Link examples to mechanics (kinematics and forces). For example, consider questions such as:

The velocity of an object is given by vector $\mathbf{v} = 3t\mathbf{i} + t^2\mathbf{j} + 4\mathbf{k}$

What is its speed after 5 seconds?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Encourage students to draw diagrams to help their geometrical thinking when answering vector questions.

Stress the importance of reading the question carefully and giving answers in the correct way, for example coordinates or column vectors may be requested.

Emphasise the importance of good notation. Students do not always understand that AP^2 represents the square of the length AP.

NOTES

Please note that vector equations of straight lines and the scalar product are *not* on this specifications.

Year 2: Remaining A Level Mathematics applied content
Section A – Statistics

Unit		Title	Estimated hours
1		**Regression and correlation**	
	a	Change of variable	**2**
	b	Correlation coefficients Statistical hypothesis testing for zero correlation	**5**
2		**Probability**	
	a	Using set notation for probability Conditional probability	**5**
	b	Questioning assumptions in probability	**2**
3		**The Normal distribution**	
	a	Understand and use the Normal distribution	**5**
	b	Use the Normal distribution as an approximation to the binomial distribution Selecting the appropriate distribution	**5**
	c	Statistical hypothesis testing for the mean of the Normal distribution	**6**
			30 hours

UNIT 1: Regression and correlation

Return to overview

SPECIFICATION REFERENCES

2.2 Change of variable may be required e.g. using knowledge of logarithms to reduce a relationship of the form $y = ax^n$ or $y = kb^x$ into linear form to estimate a and n or k and b

5.1 Understand and apply the language of statistical hypothesis testing, …., extend to correlation coefficients as measures of how close data points lie to a straight line and be able to interpret a given correlation coefficient using a given p-value or critical value (calculation of correlation coefficients is excluded)

PRIOR KNOWLEDGE

AS Mathematics – Statistics content

2.2 Understanding of regression (See Unit 2b of the SoW)
Understanding of correlation (See Unit 2b of the SoW)

AS Mathematics – Statistics content

5.1 Use appropriate language of statistical hypothesis testing (See Unit 5.1 of the SoW)

5.2 Be able to apply a hypothesis test to the binomial distribution (See Unit 5.2 of the SoW)

AS Mathematics – Pure Mathematics content

6.3, 6.4 Knowledge of logarithms (See Unit 8 of the SoW)

KEYWORDS

Hypotheses, significance level, one-tailed test, two-tailed test, test statistic, null hypothesis, alternative hypothesis, critical value, critical region, acceptance region, p-value, binomial model, correlation coefficients, product moment correlation coefficient, population coefficient, sample, inference, mean, normal distribution, variance, assumed variance, linear regression, interpolation, extrapolation, coded data

1a. Change of variable (2.2)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to change the variable in a regression line;
- be able to estimate values from regression line.

TEACHING POINTS

Start the revision of topics from year one by recapping regression.

This needs to be extended to working with changing variables (coding) within regression lines. This relies on logarithms from the pure content and students should be able to work with equations of the form $y = ax^n$ and $y = kb^x$. Students will need to know how to put these into linear form and be able to estimate a and n or k and b. An understanding of reliability when extrapolating will also need to be recapped.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Relate to real-world problems and discuss the reality of extrapolation.

1b. Correlation coefficients; Statistical hypothesis testing for correlation coefficients (5.1)	Teaching time 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand correlation coefficients;
- be able to calculate the PMCC (calculator only);
- be able to interpret a correlation coefficient;
- be able to conduct a hypothesis test for a correlation coefficient.

TEACHING POINTS

Recap scatter diagrams and the terminology used in year one to describe correlation. Students should understand that measures of correlation can be calculated to identify the strength of correlation. They need to understand that one of these, the product moment correlation coefficient (PMCC) is denoted by r, and that $|r| \leq 1$. If $r = \pm 1$ then the data points lie on a perfect straight line on a graph.

Students are expected to be able to calculate r using their calculators, but are not required to know or use the formula. They should be able to interpret their value for the PMCC in the context of the question.

Students are required to perform hypotheses tests for correlation coefficients. The hypotheses need to be stated in terms of ρ where ρ represents the population correlation coefficient. All tests should have the null hypothesis $\mathrm{H}_0\colon \rho = 0$. Tables of critical values or a p-value will be given to students.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

This is a good opportunity here to bring together the AS and A level content relating to regression and correlation.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Notation and stating a conclusion are the most common errors: ‘some students failed to state their hypotheses in terms of ρ. Common errors include failing to ensure that critical values match the alternative hypothesis and giving conclusions that do not include a reference to the context.

NOTES

A small p-value (≤ 0.05) shows strong evidence against the null hypothesis, therefore reject the null hypothesis (at the 5% significance level).

A large p-value (> 0.05) shows weak evidence against the null hypothesis, therefore accept the null hypothesis (at the 5% significance level).

UNIT 2: Probability

Return to overview

SPECIFICATION REFERENCES

3.1 Understand and use mutually exclusive and independent events when calculating probabilities Link to discrete and continuous distributions

3.2 Understand and use conditional probability, including the use of tree diagrams, Venn diagrams, two-way tables

Understand and use the conditional probability formula $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

3.3 Modelling with probability, including critiquing assumptions made and the likely effect of more realistic assumptions

PRIOR KNOWLEDGE

AS Mathematics – Statistics content

3.1 Mutually exclusive and independent events (See Unit 3 of the SoW)

KEYWORDS

Sample space, exclusive event, complementary event, discrete random variable, continuous random variable, mathematical modelling, independent, mutually exclusive, Venn diagram, tree diagram, set notation, conditional probability, two-way tables, critiquing assumptions.

2a. Using set notation for probability; Conditional probability (3.1) (3.2)	Teaching time 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use probability formulae using set notation;
- be able to use tree diagrams, Venn diagrams and two-way tables;
- understand and be able to use the conditional probability formula $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

TEACHING POINTS

Begin by recapping the use of tree diagrams and Venn diagrams, focusing on the use of set notation for probabilities. Introduce the use of two-way tables to find probabilities and use worded questions which are solved most efficiently by forming a two-way table.

Students need to be familiar with and be able to use

$\mathrm{P}(A') = 1 - \mathrm{P}(A)$,

the addition rule: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ and

the conditional probability formula $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A)$.

Use worded questions where students have to form the set notation as well as questions where the information is already given using set notation.

Ensure the teaching of this section is combined with questions to recap the properties of mutually exclusive and independent events. Make sure these are now answered using set notation too.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

With a wider probability section in the A Level content there is more scope for using real-life scenarios for probabilities. Ensure that questions are posed where it is not obvious which formulae need to be used.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Mistakes tend to involve the use of the conditional probability formula. For example wrongly assuming independence and putting $\mathrm{P}(A) \times \mathrm{P}(B)$ rather than $\mathrm{P}(A \cap B)$ as the numerator or the incorrect probability in the denominator.

2b. Questioning assumptions in probability (3.3)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to model with probability;
- be able to critique assumptions made and the likely effect of more realistic assumptions.

TEACHING POINTS

Students should know that probability can be used to predict how likely experiments are to have given outcomes. They should be able to determine all of the outcomes of an experiment (and know that these are called the sample space) and be able to determine the probability of each outcome of a given sample space. Students should also have an awareness of wider modelling where outcomes cannot be determined.

Students should be able to question and critique any assumptions made in any given scenario. For example, assumptions about independence a reasonable assumption or whether a coin or dice is fair or biased. They should be able to look at the effect of these assumptions and have an awareness of assumptions that may be more realistic.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

This sub-unit provides an opportunity for looking at real-life probability models and also for debating assumptions.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students should be careful not to make assumptions for which there is no basis. For example assuming two events are independent without having evidence or reasons for such an assumption.

UNIT 3: The Normal distribution

Return to overview

SPECIFICATION REFERENCES

4.2 Understand and use the Normal distribution as a model; find probabilities using the Normal distribution
Link to histograms, mean, standard deviation, points of inflection and the binomial distribution

4.3 Select an appropriate probability distribution for a context, with appropriate reasoning, including recognising when the binomial or the Normal model may not be appropriate

5.3 Conduct a statistical hypothesis test for the mean of the Normal distribution with known, given or assumed variance and interpret the results in context

PRIOR KNOWLEDGE

GCSE (9–1) in Mathematics at Higher Tier

A19 Solve two simultaneous equations in two variables (linear/linear or linear/quadratic) algebraically

AS Mathematics – Statistics content

3.1 Probability calculations, independent events (See Unit 3 of the SoW)

AS Mathematics – Statistics content Unit 4

4.1 Properties of the binomial distribution (See Unit 4 of the SoW)

3.1 Probability is the area under a curve (See Unit 4 of the SoW)

AS Mathematics – Statistics content Unit 5

5.1 Use appropriate language of statistical hypothesis testing (See Unit 5a of the SoW)

5.2 Be able to apply a hypothesis test to the binomial distribution (See Unit 5b of the SoW)

KEYWORDS

Binomial, discrete distribution, discrete random variable, uniform, cumulative probabilities Normal, mean, variance, continuous distribution, histogram, inflection, appropriate probability distribution.

3a. Understand and use the Normal distribution (4.2)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the properties of the Normal distribution;
- be able to find probabilities using the Normal distribution;
- know the position of the points of inflection of a Normal distribution.

TEACHING POINTS

The Normal distribution needs to be linked to histograms and the mean. A good way to introduce the topic is to look at heights on a histogram and show how it can be smoothed into the Normal distribution curve, stating this is due to the Normal being a continuous distribution.

Discuss all the properties of the Normal distribution, making sure students are confident with the symmetry of the distribution, that mean = mode = median and the asymptotic nature of the bell-shaped curve. Cover the proportions of data within 1, 2 and 3 standard deviations of the mean and remind students that the area under the curve is 1. Students are expected to know that the points of inflection on the Normal curve are at $x = \mu \pm \sigma$ (they are not expected to be able to derive this).

As with notation for the binomial distribution, students should understand the notation $X \sim \mathrm{N}(\mu, \sigma^2)$ for the Normal distribution.

Students are expected to find the probabilities from Normal distributions using their calculators. However, students do need to know the standardisation formula $Z = \frac{x-\mu}{\sigma}$ and be able to transform X values to Z values. Be clear that the denominator is the standard deviation rather than the variance which may be given.

Students should be encouraged to draw diagrams to represent the distribution and use this to check (at least for > or < 0.5) the probability they find using their calculator.

Diagrams will also help students when working backwards from a probability to find a Z value, a diagram will indicate whether the Z value is positive or negative. Again, students are expected to use their calculator to find these values.

Questions may involve the use of linear simultaneous equations to find for example both the mean and standard deviation of the Normal distribution.

You should recap the probability of independent events as this can be incorporated into questions involving the Normal distribution.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Use plenty of reverse problem examples worded in a variety of ways. Ensure students can find Z values for quartiles and percentiles.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Main errors are due to confusion between probabilities and Z values, particularly when it comes to notation, and not using the full four decimal place accuracy in calculations.

An emphasis on using diagrams alongside the calculations should help address some of the difficulties.

NOTES

Knowledge of the probability density function is not required, neither are any derivations of mean, variance or the cumulative distribution function.

3b. Use the Normal distribution as an approximation to the binomial distribution; Selecting the appropriate distribution (4.2) (4.3)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the mean and variance of a binomial distribution;
- understand and be able to apply a continuity correction;
- be able to use the Normal distribution as an approximation to the binomial distribution.

TEACHING POINTS

Begin by recapping the binomial distribution and making clear that the Normal distribution is continuous and the binomial distribution is discrete.

Students need to understand that the binomial distribution can be approximated by the Normal distribution when n is large and p is close to 0.5. Look at the parameters needed for the Normal distribution (μ and σ^2) and cover how the mean and variance are approximated from the binomial distribution ($\mu = np$ and $\sigma^2 = np(1-p)$). Students should be confident with the notation that $X \sim \mathrm{B}(n, p)$ is approximated by $Y \sim \mathrm{N}(np, np(1-p))$. Encourage them to write both distributions when answering questions involving an approximation.

When calculating probabilities for a binomial distribution which has been approximated by the Normal distribution it is important to remember that a discrete distribution has become a continuous distribution and the continuity correction needs to be introduced. It is useful here to look back at the bar chart diagrams you used in year one.

To help students understand the continuity correction label the edges of say the 8 bar with the boundaries 7.5 and 8.5 etc. If for the binomial distribution the probability $\mathrm{P}(X \leq 8)$ is required then shade the whole of the 8 bar and below; this indicates that the corresponding Normal probability is $\mathrm{P}(X < 8)$.

Using the binomial distribution, for $\mathrm{P}(X < 8)$ the 8 bar won't be shaded but every bar below it will. This indicates that using the Normal distribution the probability will be $\mathrm{P}(Y < 7.5)$. The same principle works for probabilities of the form $\mathrm{P}(X > a)$ and $\mathrm{P}(X \geq a)$. Make sure students are clear that for Normal probabilities $<$ and $\leq$ are interchangeable as it is a continuous distribution.

Once students have mastered using the Normal distribution as an approximation to the binomial distribution make sure you give them the opportunity to solve questions where they have to explain which distribution can be used before solving the problem, and whether an approximation is necessary or not. Ensure they are competent in explaining why they have chosen the distribution or approximation, clearly stating the relevant properties of their chosen distribution. They should also be able to describe why they have discounted the use of a distribution or approximation.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Use an example when n is large and p is close to 0.5 and look at a variety of cumulative probabilities in both the binomial and the Normal distributions to show students how good the approximation is to the binomial distribution.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Correctly applying continuity corrections can prove difficult with students either not applying one or otherwise adding 0.5 rather than subtracting or vice versa.

3c. Statistical hypothesis testing for the mean of the Normal distribution (5.3)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should be able to:

- be able to conduct a statistical hypothesis test for the mean of the Normal distribution;
- be able to interpret the results in context.

TEACHING POINTS

Remind students of the properties of the Normal distribution and the parameters it uses. Questions could involve a known, given or assumed variance and students should be aware of this.

Hypothesis tests need to be carried out for the mean of the Normal distribution. For $X \sim \mathrm{N}(\mu, \sigma^2)$, students need to understand that for a sample, $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.

Refer back to the formula used to translate X into Z and make sure students know they can test μ using $\dfrac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathrm{N}(0, 1^2)$.

This is the third type of hypothesis test that students are expected to be able to carry out so the importance of using the correct parameter in the hypotheses should be emphasised here. Hypotheses for the Normal distribution should be stated in terms of μ.

As in all cases conclusions need to be written clearly and in the context of the question.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Now all types of hypothesis testing have been covered students should be given mixed problems of all types from years one and two in order to practise distinguishing between tests. Ensure all hypotheses are written in terms of the correct parameter.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Common errors in exam situations include: not expressing hypotheses precisely enough; using an incorrect parameter or not using a parameter at all; incorrectly applying the continuity correction; and not giving a conclusion or answer to the question using the given context.

NOTES

Knowledge of the central limit theorem is not required, neither are proofs of the sample formulae used.

Year 2: Remaining A Level Mathematics applied content
Section B – Mechanics

Unit		Title	Estimated hours
4		**Moments:** Forces' turning effect	**5**
5		**Forces at any angle**	
	a	Resolving forces	**3**
	b	Friction forces (including coefficient of friction μ)	**3**
6		**Applications of kinematics:** Projectiles	**5**
7		**Applications of forces**	
	a	Equilibrium and statics of a particle (including ladder problems)	**4**
	b	Dynamics of a particle	**4**
8		**Further kinematics**	
	a	Constant acceleration (equations of motion in 2D; the **i**, **j** system)	**3**
	b	Variable acceleration (use of calculus and finding vectors $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ at a given time)	**3**
			30 hours

UNIT 4: Moments Forces' turning effects (9.1)	**Teaching Time** 5 Hours

Return to overview

SPECIFICATION REFERENCES

9.1 Understand and use moments in simple static contexts.

PRIOR KNOWLEDGE

- Types of forces and force diagrams
- Assumptions made throughout this course (e.g. particle, rigid, light, etc.)
- Weight = mass × gravity ($W = mg$)
- S.I. units

GCSE (9-1) in Mathematics at Higher Tier

A19 Solving linear and simultaneous equations

AS Mathematics – Mechanics content

8.4 Basic equilibrium (See Unit 8a of the SoW)

KEYWORDS

Moment, turning effect, sense, newton metre (N m), equilibrium, reaction, tension, rod, uniform, non-uniform, centre of mass, resolve, tilting, 'on the point', concurrent.

NOTES

The guidance on the specification document states: 'Equilibrium of rigid bodies. Problems involving parallel and non-parallel coplanar forces, e.g. ladder problems.'

In this unit we will be considering only horizontal rod questions in which all the forces are parallel to one another (e.g. weight and normal reaction). This unit is therefore a simple way to also introduce the concepts of resolving and equilibrium (vertically only).

Unit 7a goes on to consider applications of moments, including for example ladder problems after resolving forces at any angle has been covered in Unit 5a.

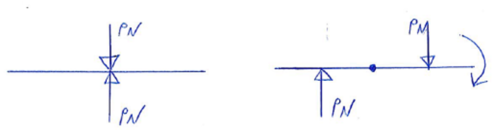
OBJECTIVES

By the end of the sub-unit, students should:

- realise that a force can produce a turning effect;
- know that a moment of a force is given by the formula force × distance giving Nm and know what the sense of a moment is;
- understand that the force and distance must be perpendicular to one another;
- be able to draw mathematical models to represent horizontal rod problems;
- realise what conditions are needed for a system to remain in equilibrium;
- be able to solve problems when a bar is on the point of tilting.

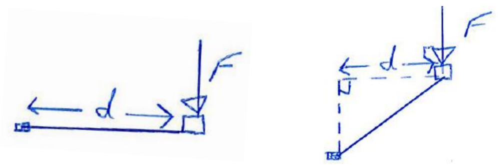
TEACHING POINTS

Start by asking two students to push up and down equally on two points of a ruler (or rod/beam) which are directly above or below each other. The forces balance and if we resolve vertically, the resultant force is zero. Hence the ruler will not move (equilibrium). However, if the two positions are separated, the ruler will turn, despite the forces still having no resultant in the vertical direction. So if two (or more) forces are not concurrent, there may be a turning effect. (See diagrams below.)



Next think about a door handle and imagine it was moved nearer the hinge of the door. Common sense tells us the door will be harder to open or close, so any formula for the turning effect of forces must involve distance as well as force.

Show a bicycle pedal in different positions and discuss which one makes turning easier. (See diagrams below.)



A discussion around this can lead to the understanding that the moment of a force, is a measure of its turning effect and is given by the formula:

moment of a force about a point = force (F) × perpendicular distance from the point to the line of action of the force (d) (the unit is newton metres, N m)

Ask students questions such as: How do we work out the distance, d, in the second bicycle pedal diagram? What additional information do we need? What if the pedal was at the topmost point, vertically above the axle?

The force and distance must be perpendicular to one another, but in this unit we will only be considering horizontal bars, supported or suspended by reactions and tensions respectively. These forces will naturally be vertical and parallel to one another, so the moments formula can be applied easily and the only thing to consider is the *sense* of the moment (whether the turning effect of each force is clockwise (negative) or anticlockwise (positive)).

Demonstrate that a uniform ruler will balance about its centre (where all the weight acts) and that this central point is therefore its centre of mass. Use this to extend students' basic idea of equilibrium as a system where there is no resultant force and also no overall turning effect, i.e. R($\uparrow$) = 0 N and the sum of the moments = 0 N m.

Make sure all the assumptions are revisited from earlier in the course e.g. model a rod as a straight line, a person standing on a bridge as a particle, strings being inextensible etc. The centre of mass is at the centre of the rod only if it is stated as being uniform.

Before starting on questions, make sure students know the notation: when we 'take moments' about a certain point (say A), we write this as M(A). Cover questions that involve:

- rods resting on two or more supports
- a rod which is suspended at two or more points
- finding the position of the centre of mass of a non-uniform rod.

Make sure you stress that theoretically we can take moments about any point and, together with resolving (vertically), we can solve any problem. However, some positions will make the solution more efficient and subsequently involve less algebra. Show students that taking moments about a point through which a force acts is zero as the distance to that force is zero.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Tilting Problems: consider rods on the point of tilting. (You could demonstrate with a ruler resting on a couple of erasers, then add some coins to the end until it lifts off one of the supports.) The forces still remain vertical and the rod horizontal (just) as the rod tends to want to lift. One of the reaction 'becomes' zero at the point of tilting.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Many students made their life more difficult than necessary by not taking the easy resolving option and using two moments equations resulting in simultaneous equations which can be difficult to solve.

Clear diagrams can help to overcome some errors such as using distances from the wrong point or missing forces (often the weight).

Students should also be reminded to read the question carefully and give their answer in the correct form – being particularly careful not to mix up weight and mass.

NOTES

This topic forms a good introduction to resolving and equilibrium as the forces are confined to only vertical examples. The later topic Equilibrium and statics (Unit 7a) deals with moments when considering ladder problems or bars held at *any* angle.

UNIT 5: Forces at any angle

Return to overview

SPECIFICATION REFERENCES

8.4 Resolving forces in 2 dimensions. Problems may be set where forces need to be resolved.

8.6 Understand and use the $F \leq \mu R$ model for friction; coefficient of friction; motion of a body on a rough surface; limiting friction and limiting equilibrium.

PRIOR KNOWLEDGE

- 2D trigonometry
- Cosine and sine rules
- $\frac{\sin x}{\cos x}$ = tan x (to find the angle of the resultant)
- Basic vectors, magnitude and direction (kinematics)
- **i**, **j** vectors
- Force diagrams and assumptions

KEYWORDS

Force, weight, tension, thrust, friction, coefficient of friction, μ, limiting, reaction, resultant, magnitude, direction, bearing, force diagram, equilibrium, inextensible, light, negligible, particle, smooth, rough, uniform, perpendicular.

NOTES

This unit is designed to help students develop the tools to enable modelling of the statics and dynamics problems in Unit 7.

The specification guidance for 8.4 states: 'Restricted to forces in two perpendicular directions or simple cases of forces given as 2D vectors.'

5a. Resolving forces (8.4)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the language relating to forces;
- be able to identify the forces acting on a particle and represent them in a force diagram;
- understand how to find the resultant force (magnitude and direction);
- be able to find the resultant of several concurrent forces by vector addition;
- be able to resolve a force into components and be able to select suitable directions for resolution.

TEACHING POINTS

Begin by considering two forces acting at right angles to one another (horizontal and vertical), use Pythagoras and trigonometry to find the hypotenuse (resultant R) and angle (direction θ above the horizontal) respectively. [You could also link to velocity from speed and vector addition rule.]

It is easy going from component form to magnitude/direction; but can we go backwards?

Guide students to consider the right-angled triangle and use trigonometry to show that the horizontal component is $R\cos\theta$ and the vertical component is $R\sin\theta$ of the Resultant, R (hypotenuse).

Show that forces given in the form **i**, **j** can be simply drawn as a right-angled triangle and the resultant and direction can be found the same way. Extend to finding the resultant of a system of forces given in **i** – **j** form by adding **i** and **j** components.

Look at two forces acting at *any* angle and show that the triangle can be solved using the cosine rule (to find the resultant) and sine rule (to find the direction).

Extend to more than two forces and resolve the system using $R(\rightarrow)$ and $R(\uparrow)$ to create two perpendicular forces, then use Pythagoras and trigonometry to calculate the resultant and direction.

Show that the weight component of a particle on an inclined plane acts in two directions: along and perpendicular to the plane. This will be a critical skill for solving the statics/dynamics questions in the next unit.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

This topic can be linked with Unit 7 and these techniques used to solve statics and dynamics problems.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When resolving common errors are: to omit g; sign errors; reversal or confusion* of when to use cos and/or sin; to omit one force (usually weight).

Students may also easily get confused by the vocabulary and mix up ‘resultant’ and ‘reaction’.

NOTES

For students who find these concepts difficult it is possible to simplify most questions by restricting the resolving of a force to using *just* cos θ. This can be done by using the method of '**cos** a**c**r**o**ss the number of degrees the force has to be turned to reach the direction we want to resolve in'.

(*This eliminates the choice of cos or sin for weaker students and can avoid the confusion mentioned in the paragraph above.)

The next section looks at the concept of friction forces, which will lead to a refined mathematical model involving rough planes.

5b. Friction forces (including coefficient of friction μ) (8.6)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand that a rough plane will have an associated frictional force, which opposes relative motion (i.e. the direction of the frictional force is always opposite to how the object is moving or 'wants' to move);
- understand that the 'roughness' of two surfaces is represented by a value called the coefficient of friction represented by μ;
- know that $0 \leq \mu$ but that there is no theoretical upper limit for μ although for most surfaces it tends to be less than 1 and that a 'smooth' surface has a value of $\mu = 0$;
- be able to draw force diagrams involving rough surfaces which include the frictional force
- understand and be able to use the formula $F \leq \mu R$.

TEACHING POINTS

Start by asking students to rub their hands together vigorously. The warmth is caused by microscopic peaks and troughs on the surface of the skin interlocking. The rougher the surface, the 'sharper' these peaks and troughs. Explain to students that this principle applies even to the smoothest looking surfaces and the force which opposes motion is called the frictional force. The value which represents the roughness is called the coefficient of friction (μ) and is zero for a smooth surface.

If we consider a book on a rough horizontal table, it will be *harder* to move the book if:-

- we put a 'paperweight' on it (increasing the reaction force)

or

- we put it on a rougher surface (increasing the value of μ).

Therefore the expression to model frictional forces uses these two factors (in direct proportion) and is given by μR. This is the maximum resistance any surface can provide before the book begins to move, so the inequality $F \leq \mu R$ applies until the force wanting to cause motion reaches the limiting value μR, called limiting friction.

Consider a 10 kg book on a rough horizontal plane. If $\mu = 0.5$, investigate the value of the frictional force if the pushing force, P is **a** 10 N, **b** 98 N, **c** 100 N

[Link to *resultant force* = *ma* from AS Mathematics – Mechanics content, see SoW Unit 8.]

Now place the book on an inclined plane and analyse the limiting friction being careful to stress that the reaction force is NOT the weight in this case. Will the book begin to slide for different angles of plane? What is the maximum angle achievable before the book slides?

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

This topic can be linked with Unit 7 and these techniques used to solve statics and dynamics problems.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students are often good at drawing force diagrams, but common errors are omitting arrowheads, incorrectly labelling (e.g. 4 kg rather than $4g$) and missing off the normal reaction or friction forces. Students can sometimes struggle to work out the direction of the frictional force.

Some students may mistakenly think that the coefficient of friction changes if the mass of an object or the angle of the slope changes.

NOTES

This topic is designed to help students develop the tools to enable modelling of the statics and dynamics problems in Unit 7.

UNIT 6: Applications of kinematics Projectiles (7.5)	Teaching Time 6 Hours

Return to overview

SPECIFICATION REFERENCES

7.5 Model motion under gravity in a vertical plane using vectors; projectiles.

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

G20 Trigonometry

AS Mathematics – Mechanics content

7.3 *suvat* formulae (See Unit 7b of the SoW)

8.3 Vertical motion under gravity (See Unit 7b of the SoW)

8.2 **i**, **j** (2D) vectors (See Unit 8a of the SoW)

AS Mathematics – Pure Mathematics content

10.1 **i**, **j** (2D) vectors (See Unit 5 of the SoW)

10.2 Magnitude and direction of a vector

5 Trigonometry (See Unit 4 of the SoW)

A level Mathematics – Pure Mathematics content

5.5 $\sec^2 x = 1 + \tan^2 x$ identity and solving trigonometric equations (See Unit 6 of the SoW)

KEYWORDS

Projectile, range, vertical, horizontal, component, acceleration, gravity, initial velocity, vector, angle of projection, position, trajectory, parabola.

NOTES

The guidance on the specification document states: ‘Derivation of formulae for time of flight, range and greatest height and the derivation of the equation of the path of a projectile may be required.’

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the time of flight of a projectile;
- be able to find the range and maximum height of a projectile;
- be able to derive formulae to find the greatest height, the time of flight and the horizontal range (for a full trajectory);
- know how to modify projectile equations to take account of the height of release;
- be able to derive and use the equation of the path.

TEACHING POINTS

Define a projectile as an object dropped or thrown in the air. Show a video from the net of a golf chip or shot-putter. Explain that the path is called a parabola which is the old Greek word for throw.

Discuss the modelling assumptions: the object is treated as a particle so it does not spin and has no air resistance. Therefore the only force on the object is gravity. (Link this back to vertical motion under gravity.)

Discuss the fact that displacement, velocity and acceleration are vectors with components in the horizontal and vertical directions. These components obey the *suvat* formulae, and the horizontal and vertical directions can be treated separately.

Begin with *horizontal* projection examples and encourage student to make two lists for the motion in the horizontal and vertical directions. It is easier to start this way as the initial *vertical* velocity is zero for this type of question, hence $u = 0$ for the vertical equation of motion.

For **all** Projectile questions:

$a = 0$ (in the horizontal direction), so the horizontal velocity is constant.

$a = 9.8\text{ m s}^{-2}$ or -9.8 m s^{-2} (in the vertical direction) depending on whether downwards is taken as positive or upwards is taken as positive.

The two equations of motion often will have time, t, as a common term.

Move on to projection with speed $U\text{ m s}^{-1}$ at *any* angle α (above the horizontal ground) and introduce the concept of the initial velocity having horizontal and vertical components. (It may be advisable to revise magnitude and direction, Pythagoras and basic trigonometry.)

Horizontally, $u = U\cos\alpha$ and if upwards is positive, vertically, $u = U\sin\alpha$.

Derive the formulae for the time of flight, greatest height (when the vertical velocity is zero) and horizontal range (for the maximum range you will need to use the identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$ which is covered in A level Mathematics - Pure Mathematics content, see SoW Unit 6d)

Emphasise the fact that s is displacement. So, for example, for the vertical equation of motion, we use $s = 0$ if the projectile returns to the ground, and if it is projected from a height and lands lower than its starting point, then, if upwards is positive s will be negative in the vertical direction.

Show examples with the initial velocity as an **i - j** vector (the **i** coefficient is u for the horizontal equation of motion). This actually makes it easier as the components are done for you.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

The general equation of the path, if we know the projection speed and the projection angle α, is a useful equation. It reduces to a quadratic (which you can show is a parabola with a negative x^2 term), but requires the identity $1 + \tan^2 \alpha = \sec^2 \alpha$ (A level Mathematics – Pure Mathematics content – see SoW Unit 6c). It can be used to find the possible angle(s) of projection to reach any point on the trajectory. There is also a symmetry of path in the absence of air resistance.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often find projectile questions challenging, sometimes confusing the horizontal and vertical aspects of the motion, for example by including the horizontal component of velocity in an equation for the vertical motion.

Other common mistakes include considering only one component of velocity when finding speeds and making sign errors when producing quadratic equations (to find t).

NOTES

General formulae can be derived to obtain the maximum height, time of flight and range in terms of initial velocity U, acceleration g and angle α, but these only work for a full trajectory. This means learning them to use in exams should be done with caution; it is probably better for students to work from first principles, rather than learn and substitute.

UNIT 7: Applications of forces

Return to overview

SPECIFICATION REFERENCES

8.2 Understand and use Newton's second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2D vectors); extend to situations where forces need to be resolved (restricted to 2 dimensions).

8.4 Understand and use Newton's third law; equilibrium of forces on a particle and motion in a straight line; application to problems involving smooth pulleys and connected particles; resolving forces in 2 dimensions; equilibrium of a particle under coplanar forces.

8.5 Understand and use addition of forces; resultant forces; dynamics for motion of a particle in a plane.

8.6 An understanding of $F \leq \mu R$ in a situation of equilibrium.

9.1 Moments: problems involving parallel and non-parallel coplanar forces e.g. ladder problems.

PRIOR KNOWLEDGE

- Types of forces and force diagrams
- Assumptions made throughout this course (e.g. particle, rigid, light, etc.)
- S.I. units
- Moments and frictional forces
- Resolving forces

AS Mathematics – Mechanics content

7	Kinematics (constant acceleration) (See Unit 7 of the SoW)
8.1, 8.2, 8.4	Newton's laws of motion (See Unit 8 of the SoW)
8.4	Basic equilibrium (See Unit 8 of the SoW)

KEYWORDS

Force, resultant, component, resolving, plane, parallel, perpendicular, weight, tension, thrust, friction, air resistance, reaction, driving force, braking force, force diagram, equilibrium, inextensible, light, negligible, particle, rough, smooth, incline, uniform, friction, coefficient of friction, concurrent, coplanar.

NOTES

The guidance on the specification document specifies: 'Problems may be set where forces need to be resolved, e.g. At least one of the particles is moving on an inclined plane.'

7a. Equilibrium and statics (including ladder problems) **Teaching time**
(8.4) (8.5) (9.1) 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand that a body is in equilibrium under a set of concurrent (acting through the same point) forces is if their resultant is zero;
- know that vectors representing forces in equilibrium form a closed polygon;
- understand how to solve problems involving equilibrium of a particle under coplanar forces, including particles on inclined planes and 2D vectors;
- be able to solve statics problems for a system of forces which are not concurrent (e.g. ladder problems), thus applying the principle of moments for forces at any angle.

TEACHING POINTS

This topic is a natural extension of AS Mathematics – Mechanics content (see SoW Unit 8a), which considers statics for systems whose forces are perpendicular (and do not need resolving at any angle) and **i**, **j** vector examples.

Recall the previous definition of equilibrium: the vector sum of the forces is zero, so the sum of their resolved parts in any direction is zero.

The book on an inclined plane provides the most common example of a weight on a slope. Stress the importance of key phrases like 'rough plane', which will introduce a frictional force. Also highlight the part of the sentence that says 'the book is *on the point* of moving *down* the plane' and emphasise that this indicates that the frictional force is in the up direction and is at its limiting value.

Cover examples

- Where the angle of incline is given in arctan or arcsin form, so students have to construct and read off sin and cos of the angle.
- Where weights are held in equilibrium by two strings at any angle (this is the same as a weight being tied onto a particular point of a single string – the knot makes it effectively two pieces of string with two different tensions).You could show an alternative graphical solution. For example, combining. the three forces to form a closed triangle (equilibrium means no resultant). Applying the sine rule to this triangle gives a useful result called Lami's theorem, but it can only be used for three forces in equilibrium.
- Where a ring is free to slide on a string (hence one tension).
- Where the forces are given in terms of **i** and **j**.

Finally, move on to ladder-type problems which will revise moments and then extend to any angle, as the forces will not be concurrent. Extend the moments formula to '*perpendicular force* × *distance*' and resolve the force to find its component at right angles to the full distance from the moments point.

Show students how to use the alternative formula '*force* × *perpendicular distance'*, by measuring the perpendicular distance from the moments point to the line of action of the force.

Also make sure that students are clear about the directions of the frictional force (for examples involving rough surfaces) and the reactions at the wall and ground being labelled differently.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Extension: consider a uniform rod which has one end *freely hinged* to a wall and the other end tied to a point above the wall, making the bar horizontal. Discuss the fact that the reaction at the hinge is *not* perpendicular to the wall and that the lines of actions of all the forces in the system will all meet at one point for equilibrium. Representing the reaction at the hinge as two perpendicular forces, the 'resolving and taking moments' solution would be fairly straightforward.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students are often good at drawing force diagrams, but common errors are omitting arrowheads, incorrectly labelling (e.g. 4 kg rather than $4g$) and missing off the normal reaction or friction forces. Students can sometimes struggle to work out the direction of the frictional force.

Common errors in questions involving moments are ignored the weight of the ladder, sine/cosine confusion and missing a distance in one or more terms.

NOTES

The guidance on the specification document states 'Problems may be set where forces need to be resolved. (Restricted to forces in two perpendicular directions or simple cases of forces given as 2D vectors.)'

7b. Dynamics of a particle (8.2) (8.4) (8.5) (8.6)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know and understand the meaning of Newton's second law;
- be able to formulate the equation of motion for a particle in 1-dimensional motion where the resultant force is mass × acceleration;
- be able to formulate the equation of motion for a particle in 2-dimensional motion where the resultant force is mass × acceleration;
- be able to formulate and solve separate equations of motion for connected particles, where one of the particles could be on an inclined and/or rough plane.

TEACHING POINTS

This topic is a natural extension of AS Mathematics – Mechanics content (see SoW Unit 8a), which considers dynamics for systems whose forces are perpendicular (and do not need resolving at any angle) and **i**, **j** vector examples.

Recall the previous definition of dynamics: the *vector sum of the forces* = *mass* × *acceleration*, so the sum of their resolved parts in any direction can now be represented as a *single* force. This force is called the resultant and is equal to *mass* × *acceleration* (Newton's second law).

We can use the equations of motion for constant acceleration to describe the motion in more detail e.g. time taken to come to rest etc.

The basic mathematical modelling is identical to that of setting up a statics problem, except when you resolve in the direction of motion; there will be a 'winning' resultant force.

For inclined plane problems stress, that it is often easier, to resolve along and perpendicular to the plane. Some students find it hard to understand that even though the particle is moving up/down, the forces are 'balanced' if we resolve perpendicular to the plane.

Make sure you cover examples in which a force 'pushing' up the plane is removed at a certain point. This means the frictional force and component of weight now influence the subsequent motion and act as 'braking forces' causing a retardation, bringing the particle to instantaneous rest (and then the friction changes direction, as the particle wants to slide back down the plane).

Provide some examples where the forces are given in terms of **i** and **j**. These are solved by applying Newton's Second Law in vector form, hence $\boldsymbol{F} = m\boldsymbol{a}$.

Connected particle problems (previously covered in AS Mathematics – Mechanics content, see SoW Unit 8b) can now be extended so at least one of the particles is placed on a rough or smooth inclined plane and/or a rough horizontal plane. This introduces the resolving and frictional concepts from the previous unit.

For 'car and caravan' type questions, the tow rope or tow-bar can now be modelled at an angle rather than horizontally.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

To make this dynamics topic more real, you could set up experiments, involving connected particles for example, and video the motions. You can then see that when one particle hits the ground, the second particle continuing to move up and the string become slack.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors candidates make include: confusing the terms 'resultant' and 'reaction'; incorrectly treated the scenario as a statics problem and assuming the forces are in equilibrium; omitting g from the weight term; and, more rarely, including g in the 'ma' term.

NOTES

The guidance on the specification document states 'Connected particle problems could include problems with particles in contact e.g. lift problems.'

Students may be required to resolve a vector into two components or use a vector diagram, e.g. problems involving two or more forces, given in magnitude–direction form.

UNIT 8: Further kinematics

Return to overview

SPECIFICATION REFERENCES

7.3 Extend the constant acceleration formulae of motion to 2 dimensions using vectors.

7.4 Use calculus in kinematics for (variable acceleration) motion in a straight line. Extend to 2 dimensions using vectors.

PRIOR KNOWLEDGE

- Basic trigonometry, Pythagoras and vectors
- Find the magnitude and direction of vectors

AS Mathematics – Mechanics content

7 Kinematics 1 and equations of motion (See Unit 7b of the SoW)

7 Kinematics 2 (variable force) (See Unit 9 of the SoW)

AS Mathematics – Pure Mathematics content

10.1 2D vectors – **i**, **j** system (See Unit 5 of the SoW)

KEYWORDS

Distance, displacement, speed, velocity, constant acceleration, constant force, variable force, variable acceleration, retardation, deceleration, initial (t = 0), stationary (speed = 0), at rest (speed = 0), instantaneously, differentiate, integrate, turning point.

NOTES

This topic builds on the kinematics covered in AS Mathematics – Mechanics content, see SoW Units 7 and 9.

8a. Constant acceleration (equations of motion in 2D; the i, j system) (7.3)	Teaching time 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to recognise when the use of constant acceleration formulae is appropriate;
- be able to write positions, velocities and accelerations in vector form;
- understand the language of kinematics appropriate to motion in 2 dimensions
- be able to find the magnitude and direction of vectors;
- be able to extend techniques for motion in 1 dimension to 2 dimensions by using vectors;
- know how to use velocity triangles to solve simple problems;
- understand and use *suvat* formulae for constant acceleration in 2D;
- know how to apply the equations of motion to **i**, **j** vector problems;
- be able to use $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$, $\boldsymbol{r} = \boldsymbol{u}t + \frac{1}{2}\boldsymbol{a}t^2$ etc. with vectors given in **i** , **j** or column vector form.

TEACHING POINTS

This topic enables us to use the familiar *suvat* formulae for constant acceleration for more complex motions in two dimensions. It is important to stress that the acceleration may have different values for the **i** and **j** components, but is fixed in value for that direction and is therefore constant. Illustrate this by reviewing projectile motion (covered in Unit 6), which showed that the acceleration was zero in the horizontal direction and $\pm 9.8\text{ m s}^{-2}$ in the vertical direction, hence for a full trajectory $\boldsymbol{a} = (0\mathbf{i} - 9.8\mathbf{j})\text{ m s}^{-2}$. This gives a curved (parabolic) path even though the accelerations are constant.

Cover examples which ask for the speed, distance and direction of motion. Make sure that students can pick out the keywords, and realise when the answer can be left in **i**, **j** form and when to form a triangle and use Pythagoras and tan to calculate the magnitude and direction (e.g. when asked for the speed and direction of motion of a particle).

Also stress that the *angle* of the velocity vector gives the true direction of motion and that the acceleration's magnitude does not have a special keyword, but will just be asked for as magnitude of the acceleration.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Projectile questions could also be tackled using the vector equations of motion rather than separating out the horizontal and vertical motions (Unit 6).

Some graphical packages will draw the graphs using **i - j** vectors; these can be used to help students visualise the problems.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Candidates are generally able to use *suvat* equations in 2D to find unknown heights, velocities etc. However, some common errors are: finding a solution in vector form and not extracting one component e.g. to find the height; incorrectly finding velocity rather than speed and vice versa; and equating scalars and vectors and forgetting to split e.g. velocities into **i** and **j** components.

NOTES

If there is change in motion, we have a dynamics problem. These are solved by applying Newton's second law in vector form: $\boldsymbol{F} = m\boldsymbol{a}$. This naturally leads onto the next section in which the force is variable.

8b. Variable acceleration (use of calculus and finding vectors $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ at a given time) (7.4)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to extend techniques for motion in 1 dimension to 2 dimensions by using calculus and vector versions of equations for variable force/acceleration problems;
- understand the language and notation of kinematics appropriate to variable motion in 2 dimensions, i.e. knowing the notation $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ for variable acceleration in terms of time.

TEACHING POINTS

This topic links directly to, and is an extension of AS Mathematics – Mechanics content (see SoW Unit 9), which used:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t},\ a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2} \text{ and } s = \int v\,\mathrm{d}t,\ v = \int a\,\mathrm{d}t$$

to model the rates of change for motion of a particle subject to a variable force.

Motions can now be more complicated as the forces in the **i** and **j** directions can differ and be variable (i.e. $\boldsymbol{F} = m\boldsymbol{a}$). Also the notation for 2D motion replaces the displacement, s, with position vector, $\boldsymbol{r}$. Velocity, $\boldsymbol{v}$, can be defined as $\dot{\boldsymbol{r}}$ and the acceleration vector can be called $\ddot{\boldsymbol{r}}$ (rather than $\boldsymbol{a}$).

Introduce this notation to students, explaining how the dot above the $\boldsymbol{r}$ denotes how many times the $\boldsymbol{r}$ has been differentiated with respect to time. Hence $\ddot{\boldsymbol{r}}$ (representing the acceleration) effectively means $\boldsymbol{r}$ differentiated *twice* with respect to time or $\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$.

The other vital point to stress is when we integrate $\dot{\boldsymbol{r}}$ (or $\boldsymbol{v}$) to obtain the displacement $\boldsymbol{r}$, we have to introduce a vector constant of integration in the form $c\mathbf{i} + k\mathbf{j}$ (rather than just + c). Any conditions provided in the question (e.g. the particle is initially at the point with position vector $(3\mathbf{i} + 2\mathbf{j})$ m) allow us to substitute into the expression for $\boldsymbol{r}$ and calculate the constants.

Ask questions along the lines of:

Consider an aeroplane taking off. Its position is given by $\boldsymbol{r} = (80t\,\mathbf{i} + 0.5t^3\mathbf{j})$ m. What is its velocity and acceleration at time t? Now criticise the model. (Hint: consider motion in the x-direction)

Reverse the process: a particle has acceleration $\boldsymbol{a} = (4t\mathbf{i} + 2\mathbf{j})$ m s^{-2} and is initially at the origin moving with velocity $2\mathbf{i}$ m s^{-1}. Find $\dot{\boldsymbol{r}}$ and $\mathbf{r}$ using integration. (Be careful with the constants of integration!)

Just as in the 1-dimensional case, we do not need to use calculus every time; if the acceleration vector is constant, we can use vector forms of the *suvat* formulae as in Unit 8a.

Questions on this topic often ask about the direction of motion: stress that this is given by the direction of the velocity vector. To find when an object is moving due North, the East component of the velocity vector is zero and the North component positive.

Finally, a question may ask for the force acting on the particle of mass m kg. In this situation students will need to find the acceleration ($\ddot{\boldsymbol{r}}$) at time t and then state the force $\boldsymbol{F}$ as $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ or $\boldsymbol{F} = m\boldsymbol{a}$ (in terms of **i** and **j**).

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

The variable is always t for this unit. (See Further Mathematics – Further Mechanics 2 content, for when the force is dependent on other factors and variables such on the displacement or velocity.)

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some common errors students make include: forgetting the constant of integration; giving the final answer as a vector when the question asked for the speed; and not being careful about changes of direction and so, for example, finding the displacement rather than the distance travelled.

NOTES

The following diagram may help students decide whether to differentiate or integrate to solve a problem. '**d**' for the **d**own arrow means '**d**ifferentiate'. Hence, down from $\boldsymbol{r}$ gives $\boldsymbol{v}$ or $\dot{\boldsymbol{r}}$ or $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = v$. Integration is the opposite of differentiation so up is integrate. Up from $\boldsymbol{a}(\ddot{\boldsymbol{r}})$ gives $v(\dot{\boldsymbol{r}})$ or integral of $\boldsymbol{a}(\ddot{\boldsymbol{r}})$ with respect to t gives $\boldsymbol{v}(\dot{\boldsymbol{r}})$.

$$\begin{array}{ccc} & \downarrow \boldsymbol{s} & \\ \textbf{d}\text{iff} & \downarrow \boldsymbol{v}\,(\dot{\boldsymbol{r}}) \uparrow & \text{int} \\ & \boldsymbol{a}\,(\ddot{\boldsymbol{r}}) \uparrow & \end{array}$$

Acknowledgment: Screenshots of the videos which can be found throughout A level Mathematics: Pure Mathematics section of this scheme of work (framed colour diagrams with annotations) have been used by permission and thanks to ExamSolutions - Maths Revision Made Easy.

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